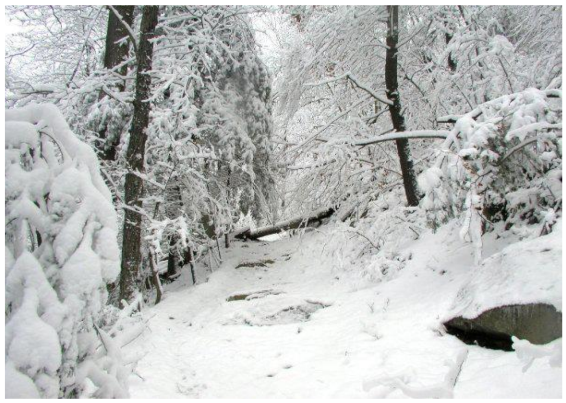
Modern Optics, Prof. Ruiz, UNCA doctorphys.com
Chapter L. Waves, Phasors and Packets

L0. Introduction. You have experienced the power of physics in many interdisciplinary applications. Let's review this wealth of connections. The critical thinking skills you are enhancing and further developing in this course will serve you well in any future studies and employment. Ten disciplines are listed below.

- ATMS, Atmospheric Science (Meteorology) – mirages and rainbows
- ASTR, Astronomy – telescopes, solar spectrum, eclipses
- BIOL, Biology – the optics of the human eye (cornea and eye lens)
- CHEM, Chemistry – light & electron transitions (Hydrogen, Sodium, Mercury, Cadmium)
- MAG, Magic – illusions based on optics
- MATH, Mathematics – algebra, geometry, trigonometry, and calculus
- ENGR, Engineering – design of mirrors (wide-angle, vanity) and lenses
- MED, Medicine – visual acuity, prescribing eyeglasses for myopia and hyperopia
- PHOT, Photography – f/#, aperture, camera lenses
- PHYS, Physics – Laws of Reflection, Refraction, Spherical Mirrors, and Lenses

This chapter will focus on connections within physics itself. Think of the chapter as taking a tour through a landscape of physics related to waves, similar to taking a hike, observing on a trail.

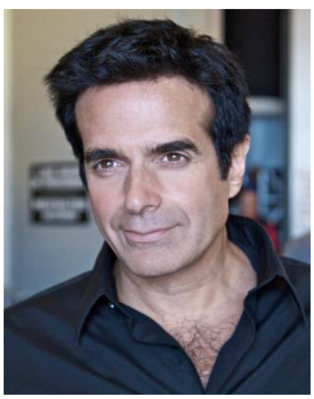
Photos by Doc, March 2, 2009, Trail Near Home, Beaverdam

Most of this chapter will be your going on a tour will me and I will be your tour guide. The landscape with be theoretical physics. To set a good example, my tour will be off the top of my head to show you a unique way of integrating the basics of wave physics. I relied on my teachers years ago and those that inspired me the most were original in their teaching. I could tell that these teachers had incorporated the physics into themselves and were giving me their unique and original explanations – as the tour guide. Later, hopefully, you can give you own tours of physics landscapes, as you will begin to see features that others do not see.

Richard Feynman (1918-1988)
Theoretical Physicist
Courtesy nobelprize.org

Richard Feynman was an excellent example of a theoretical physicist. He worked everything out for himself, often discovering new and more powerful ways of doing the math and physics.

Feynman once stressed the importance of knowing more than one way to understand a physics issue. He was giving a talk in 1964 and was referring to good theorists and representations of a specific physics problem. I would like to recommend that physics students try to understand some physics of their choice with 4 or 5 different representations. Feynman was so clever that he was considered a wizard or magician at theoretical physics.



David Copperfield (b. 1956)
Magician and Illusionist
Courtesy Homer Liwag, Released into the Public Domain

I quote the magician David Copperfield, where a similar comment was made regarding magic secrets.

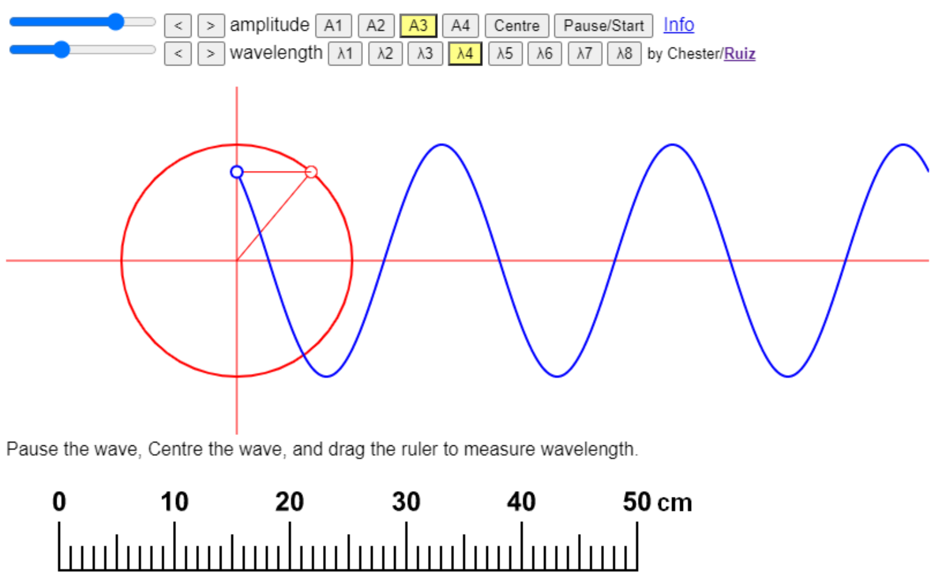
"I don’t think people want to know. I don’t think people really care about knowing. They enjoy the fantasy, you know. In my career there’s been so many people making guesses, and that’s part of the fun, I guess, for a certain percentage of the audience, trying to guess how things work. I have four or five methods for each of the illusions, and I keep changing the methods.

"It’s not unlike what Houdini does. He had many ways of escaping from a jail cell, not just one way. He had many different ways. So if people were kind of onto him, or if he wrote a book about how he did it, there was many other ways of doing it. And the net result is, it’s not about the secret. It’s not about how it’s done. It’s the feeling of being able to do it. It’s the wonder that’s created by the act itself." ***David Copperfield*** (www.pbs.org)

David Copperfield even looks a little like Feynman. Both are artists of the highest caliber in their own right. Feynman shared the 1965 Nobel Prize in Physics and Copperfield according to *Forbes* “is the most commercially successful magician in history.” Forbes, Houdini in the Desert, 2006

L1. The Circle and the Sine Wave. The connection between motion along the circumference of a circle and the sine wave "can be traced back to Newton," (quoted from my paper listed here).

Ethan Chester (Asheville High School Student) and Michael J. Ruiz, "HTML5 lab app relating circular motion to harmonic motion and the wave equation," *Physics Education* **55**, 013004 (January 2020). [Click for the App]



The application circleSine developed by Ethan Chester and me.

Coauthor Ethan was a high-school student at the time. He had contacted then Chair Professor Bennett and asked to do an intern at UNCA just for the experience. I took him on when I found out he could program. I wanted him to get something more than just experience, therefore I configured a computer project for him that was publishable. So we can add another interdisciplinary area here:

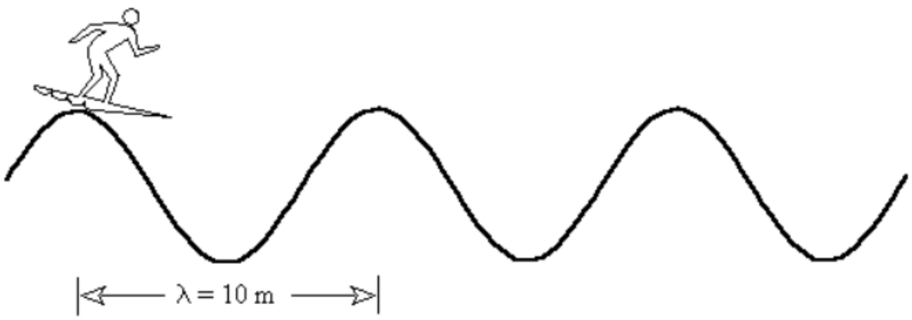
- CSCI, Computer Science – creating a computer visualization for the sine wave.

We were not the first to accomplish such a visualization. But remember, my Art major student Halima Flynt was not the first to make a camera obscura. However, if you add a unique innovative slant to a topic, it may be publishable. You can drag the ruler in our app, measure wavelengths, and adjust amplitude, frequency and wavelength. The app can be used in an online lab, like we do for my *PHYS 102 The Physics of Sound and Music*. We made it free for non-commercial use.

One advantage of computer science is that you can program a device that can be impossible to actually make and distribute as real equipment. The visualization provided here can give insight into waves. Such motion that results in sine waves is called **harmonic motion**. From the app you can see that shorter wavelength means higher frequency, the amplitude does not affect the frequency or wavelength, and the speed of the waves to the right is independent of the amplitude, wavelength, and frequency. Frequency and wavelength are related to the wave speed as

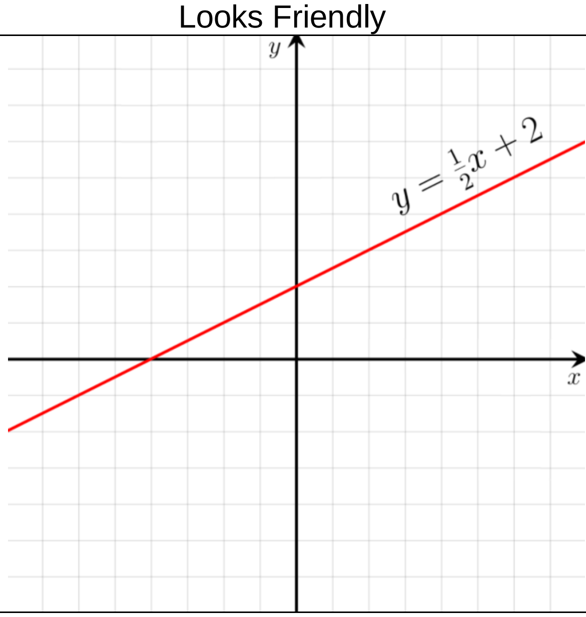
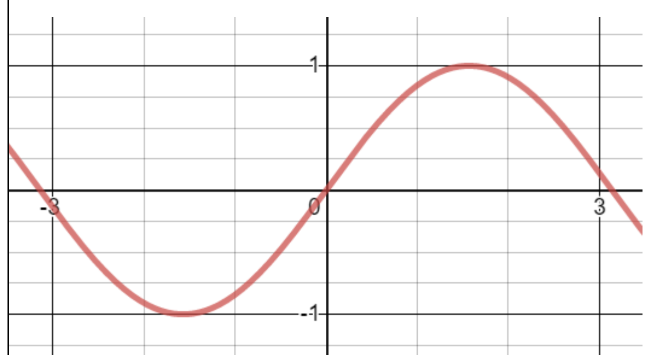
$$v = \lambda f \ .$$

The speed depends on the property of the medium. For a nondispersive medium, all wavelengths travel at the same speed. Here is my ideal surfing figure that I use in my liberal arts intro courses. If the wavelength is 10 m and 5 crests go by per second, what is the speed? They say 50 m/s. I write down 10 m times 5 1/s and then replace the friendly numbers with the appropriate symbols.



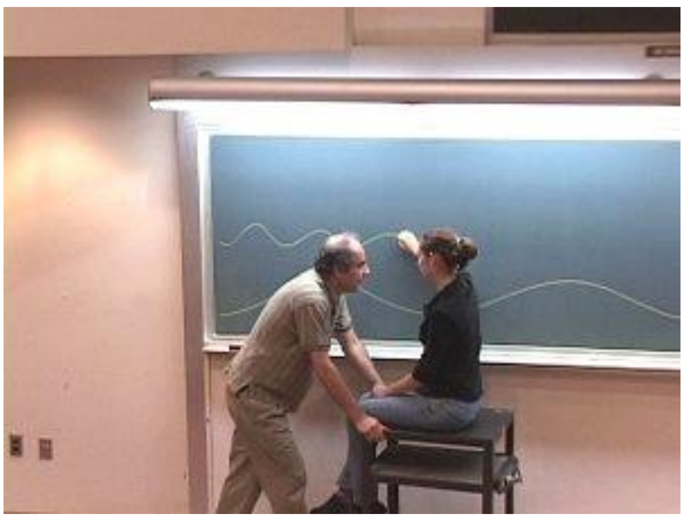
L2. The Ubiquitous Sine Wave.

When I was in high school, I though the sine wave was complicated. After all, isn't y = mx + b the easiest function ever? But it turns out that the sine wave is nature's wave – and simple too.

Looks Friendly	Looks Scary
	
Courtesy Jim.belk Released to the Public Domain	Courtesy www.desmos.com

Making Sine Waves in Class.
PHYS 102 The Physics of Sound and Music,
August 20, 2015.
Still from Video by Lara Fetto.

Here we have the “human oscilloscope.” A student waves up and down naturally, looking away from the blackboard so as not to intentionally draw anything. Two students push the cart to sweep out the trace like an oscilloscope.



Doc and daughter Christa.
Filming a demonstration in 2001.
Still from Video by Evan Ruiz.

Note the sine wave on the board, When Christa waves at a higher frequency (the top wave), the wavelength is shorter.

Therefore the demonstration also provides for a visualization of the wave formula

$$v = \lambda f \ .$$

The sine wave is nature’s wave. People wave sine waves. The small oscillations of a pendulum and a mass attached to a spring with no resistance produce sinusoidal motion. Atoms vibrate in a sine fashion. Whenever a charged particle vibrates it shakes off a light wave. Why is the sine wave everywhere? To answer this question we turn to potential energy.

In introductory physics the potential energy for gravity near the Earth’s surface is

$$V(z) = mgz \ .$$

The force is given by negative the derivative:

$$F = -\frac{dV(z)}{dz} = -mg \ .$$

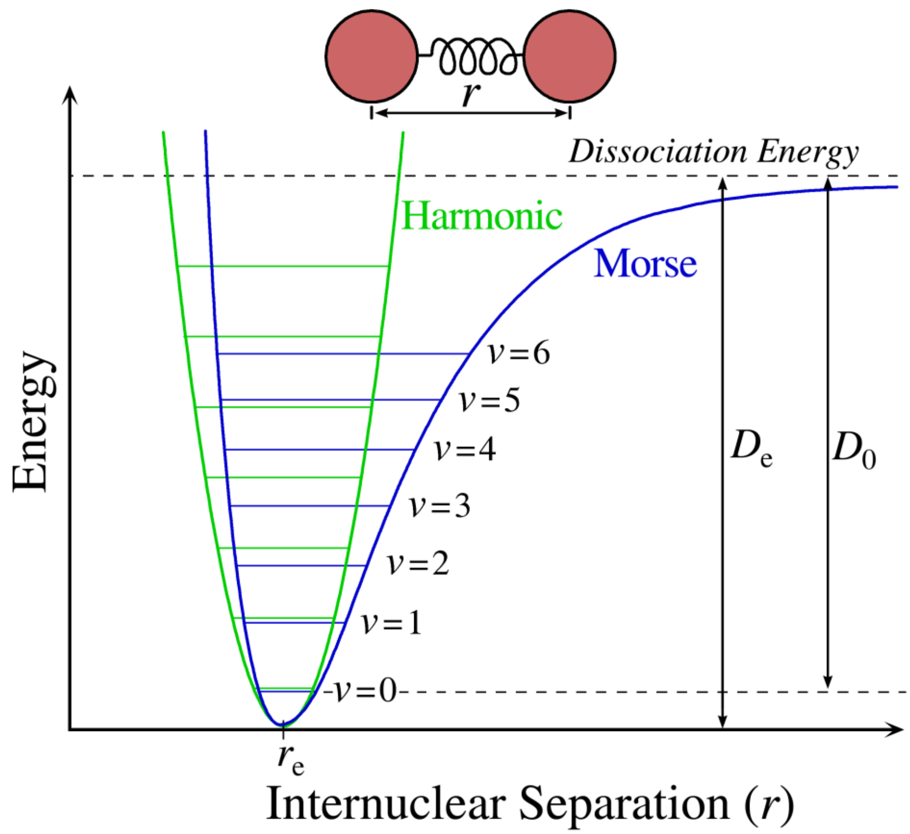
For the spring

$$V(z) = \frac{1}{2}kx^2,$$

$$F = -\frac{V(x)}{dx} = -kx,$$

which is Hooke's Law.

Now consider some complicated potential that has a stable equilibrium point, i.e., a minimum. How about the Morse potential that describes a diatomic molecule? Here is another chemistry connection in our course



Wikimedia: Mark Somoza
Creative Commons License

We will move the coordinate axes so that (x,E) = (0,0) at the minimum. Note E = V.

Then we can apply the Taylor series expansion of a power series for V(x).

Remember back in your days of calculus, the following general formula?

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

See the review below.

$$f(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f^{(3)}(0)x^3 + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

$$f^{(n)}(0) \equiv \left[\frac{d^n}{dx^n}f(x)\right]_{x=0}$$

Applying the power series to the potential function, which I emphasize, can have any shape as long as it has that minimum so we have a stable situation somewhere.

$$V(x) = V(0) + V^{(1)}(0)x + \frac{1}{2!}V^{(2)}(0)x^2 + \frac{1}{3!}V^{(3)}(0)x^3 + \ldots$$

Small x => $V(x) \approx V(0) + V^{(1)}(0)x + \frac{1}{2!}V^{(2)}(0)x^2$

At a minimum => $V(x) \approx V(0) + \frac{1}{2!} f^{(2)}(0)x^2$ since the slope is zero there.

We may always choose $V(0) = 0$, the reference.

$$V(x) \approx \frac{1}{2!}V^{(2)}(0)x^2$$

Hooke's Law: $F = -\frac{V(x)}{dx} = -V^{(2)}(0)x = -kx$

$$k = V^{(2)}(0)$$

What? All potentials that have stable extrema, i.e., minima, approximate Hooke's Law for small oscillations there. We should solve the Hooke's Law problem. Guess what the solution will be? Our sine wave friend. Watch!

$$F = -\frac{V(x)}{dx} = -kx$$

$$F = ma = m\ddot{x}$$

$$m\ddot{x} = -kx$$

$$\ddot{x} = -\frac{k}{m}x$$

Which function do you know when you take two derivatives you get it back with a minus sign?

$$x(t) = \sin(\omega t)$$

$$\dot{x}(t) = \omega\cos(\omega t)$$

$$\ddot{x}(t) = -\omega^2 \sin(\omega t)$$

$$\omega^2 = \frac{k}{m}$$

$$\omega = \sqrt{\frac{k}{m}}$$

We expect a second general solution since we have a second order differential equation.

$$x(t) = \cos(\omega t)$$

$$\dot{x}(t) = -\omega A\sin(\omega t)$$

$$\ddot{x}(t) = -\omega^2 A\cos(\omega t)$$

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

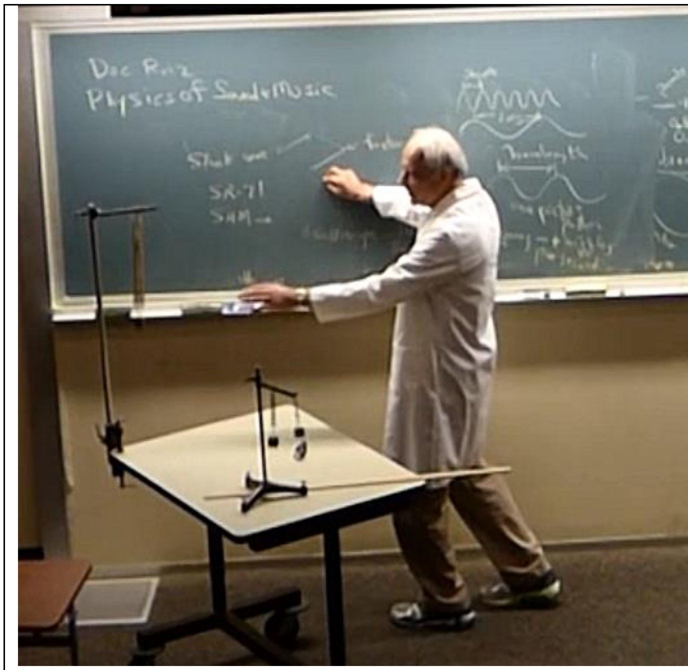
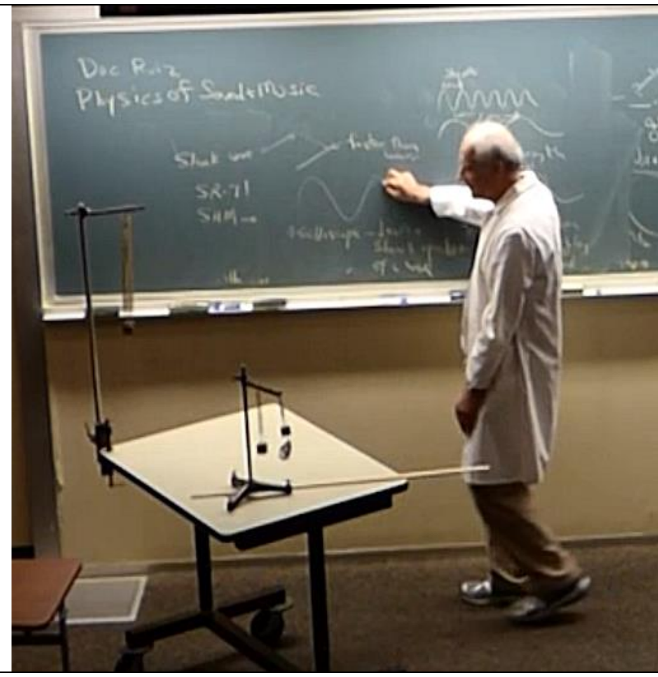
Initial conditions => A, B.

Pull back and let go => B = 0

Whack it => A = 0

Below is how I solve the differential equation for non-science majors and grade-schoolers. I follow the mass on the spring with my left hand and let my right hand go up and down in sync as I back up to the right. The result is the sine wave solution on the blackboard.

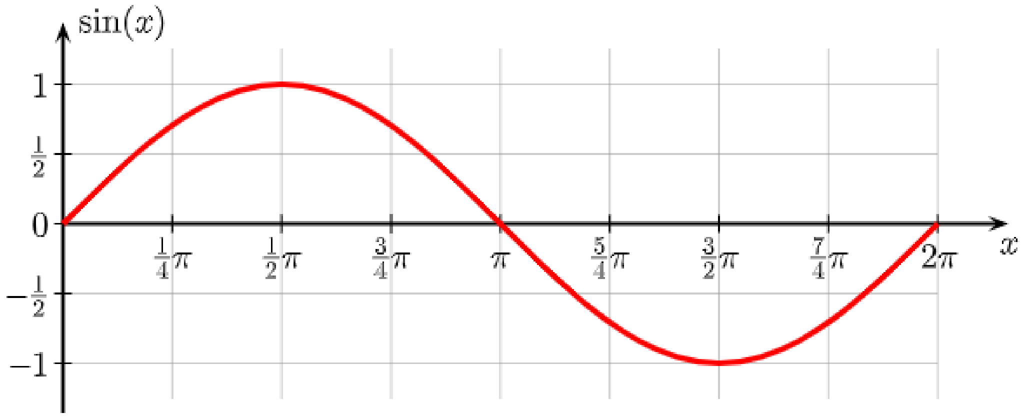
I directly write the solution on the board following the motion. Such motion is called **simple harmonic motion**. And all stable minima will exhibit sine wave oscillations for small displacements from equilibrium!

	
Follow oscillating mass with left hand	Right hand mimics left to draw sine wave.

Now you can see why the sine wave is nature's wave! And shows up everywhere!

L3. Adapting the Sine Function for Waves.

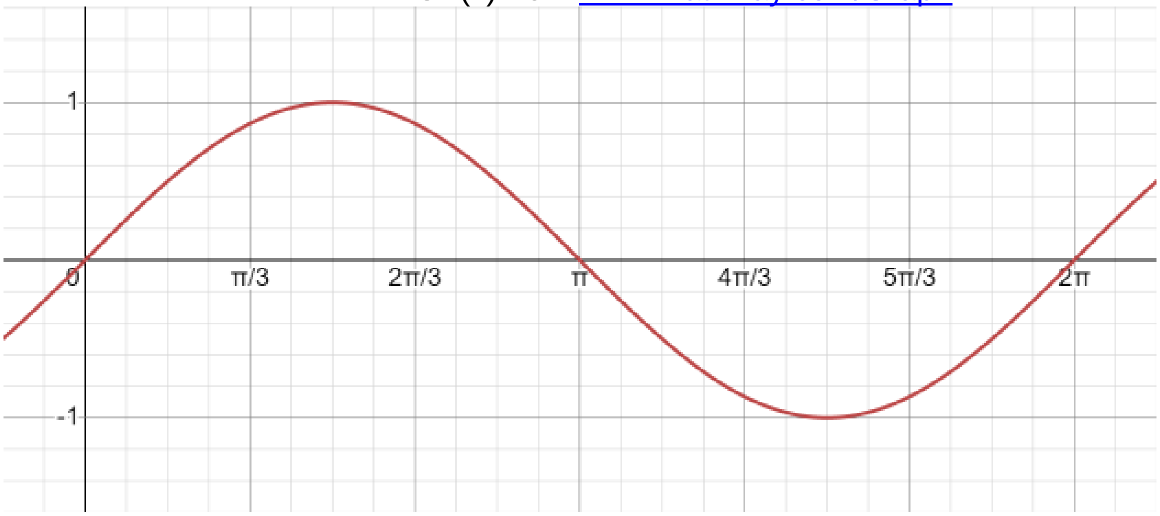
The sine function is shown below in its purely mathematical form. We need to prepare the sine wave for use in physics, where we have concepts such as wavelength and frequency.



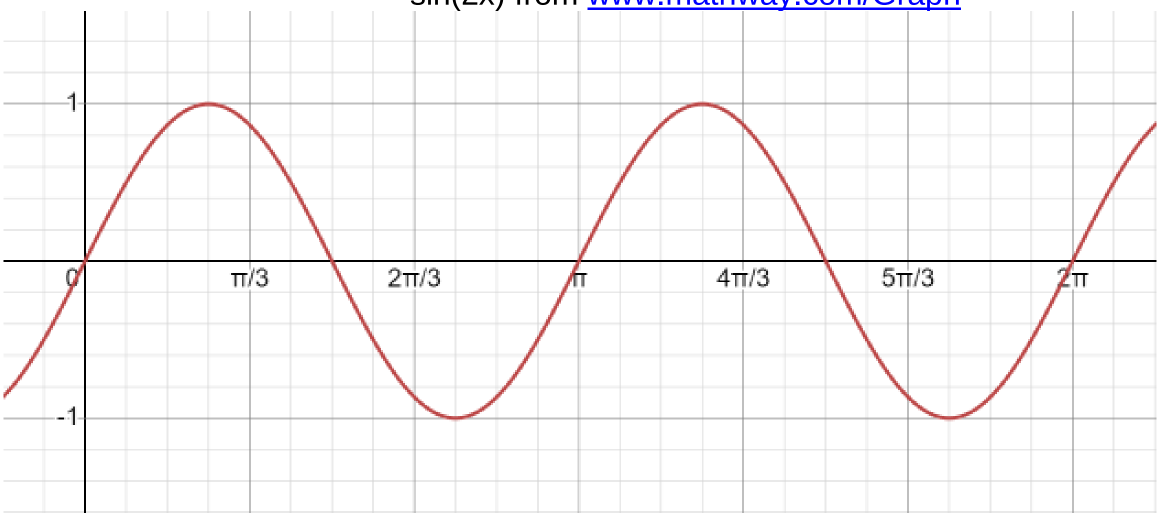
Wikipedia: Geek3. Public Domain.

See below for what happens when we compare with sin(2x) and sin(3x).

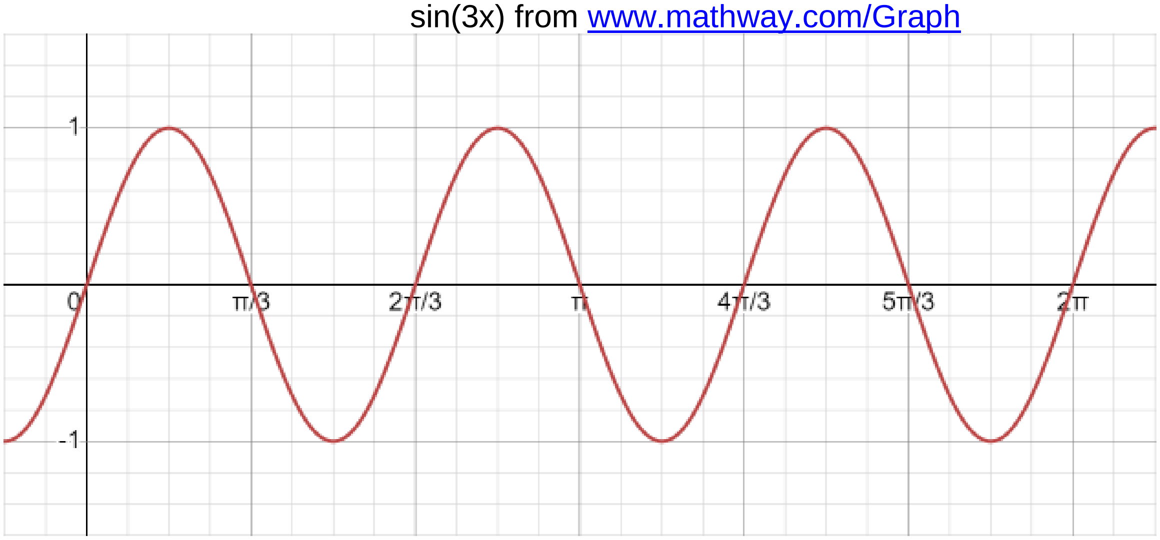
sin(x) from www.mathway.com/Graph



sin(2x) from www.mathway.com/Graph



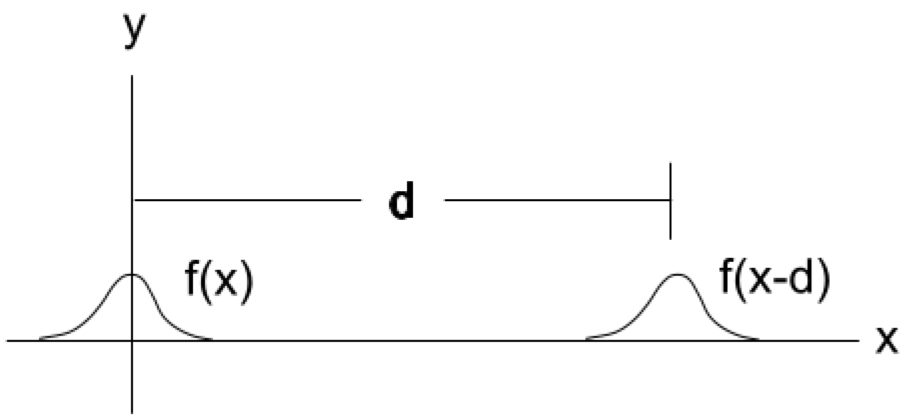
sin(3x) from www.mathway.com/Graph



For sin(nx) there are n wavelength that fit between 0 and 2π. However, we need sin(kx) as our basic expression since in physics we want x to be meters, feet, i.e., something with the dimensions of length. So the "k" is needed to convert to a dimensionless quantity that the sine function can then work on. In summary, kx will be dimensionless, like radians.

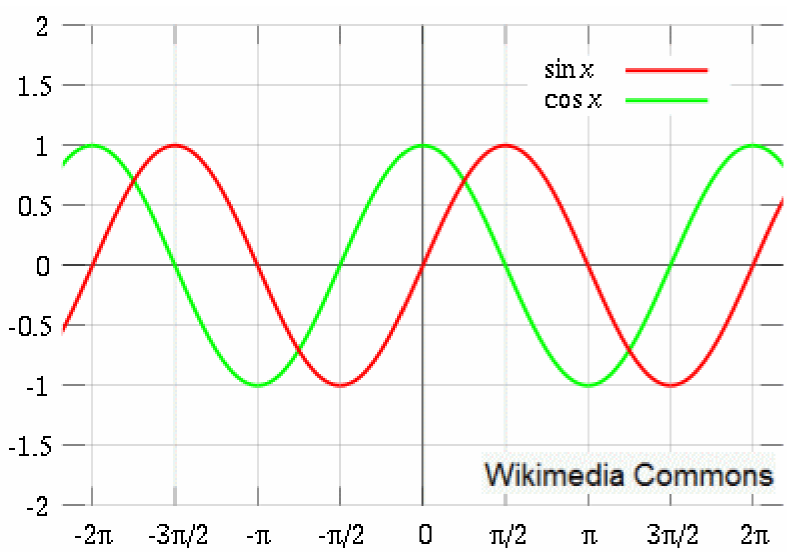
So far we have sin(kx). We can easily add amplitude: A sin(kx). Now we need to add motion to the wave so that it can travel. We start by looking into what shifts a function to the right.

A function $f(x)$ is shown with a peak at f(0). Denote this by writing $f(0) = peak$. If we shift this function to the right by a distance d, then the new function $h(x)$ must be $h(x) = f(x-d)$. Here is how you can check this rule. Is the peak now at $x = d$? Does $h(d) = peak$? We work out the details below the figure.



$$f(0) = peak \quad \text{and} \quad h(x) \equiv f(x-d)$$

$$h(d) = f(d-d) = f(0) = peak$$



It checks out. Do you remember doing this often in trigonometry? If you shift the cosine by $\pi/2$ to the right, you get the sine.

$$\sin x = \cos(x - \frac{\pi}{2})$$

The above relation also tells you that the sine of an angle in a right triangle equals the cosine of its complement.

Since $f(x-d)$ is our shifted function to the right by a distance d, we can let $d=vt$ to obtain a traveling function to the right. Common practice is to use ψ for a wave. So we write

$$\psi(x,t)=f(x-vt).$$

For a wave traveling to the left, switch the sign in front of v.

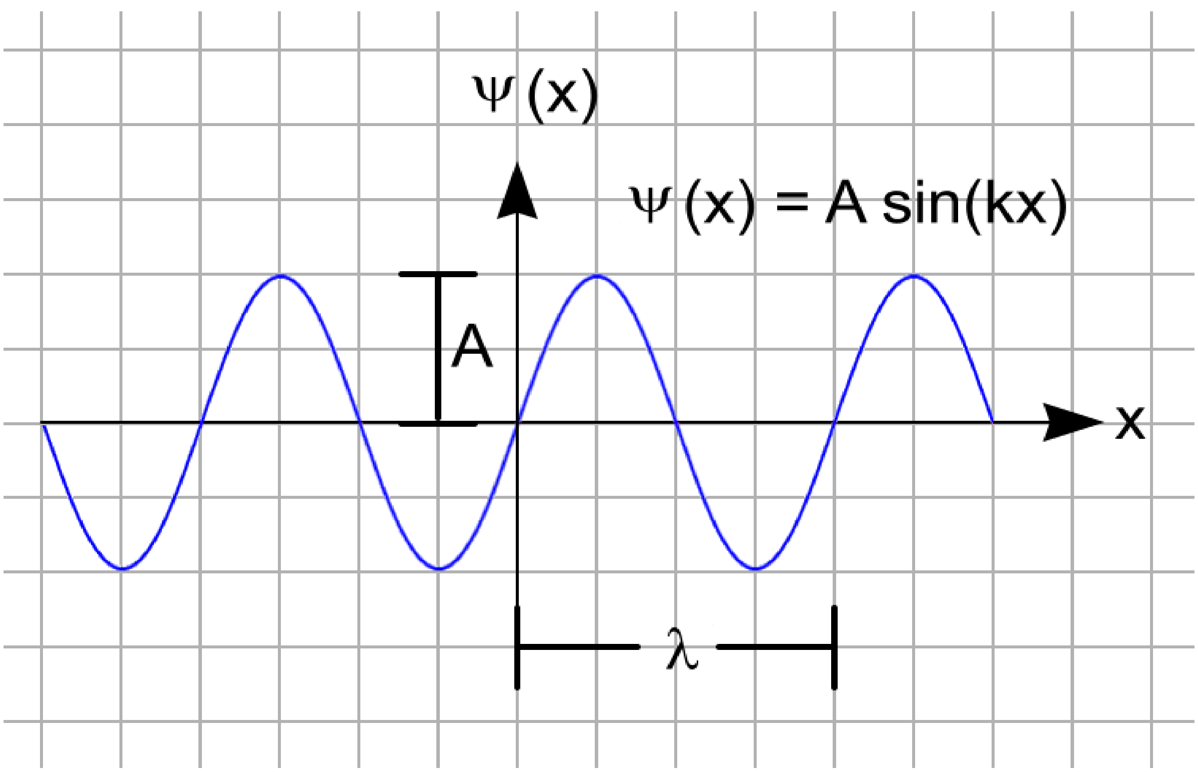
$$\psi(x,t)=g(x+vt).$$

Applying these concepts to $f(x)=\sin(kx)$, we obtain

$$\psi(x,t)=\sin\left[k(x-vt)\right].$$

Pick t = 0 to freeze the wave at a given instant in time so we can analyze sin(kx) from the point of view of physics. Then, with the inclusion of the important physical quantity – the amplitude A,

$$\psi(x)=A\sin(kx).$$



Wavelength λ and amplitude A are important physical quantities in physics. We are dressing up the math to describe the physics of waves.

One wavelength is realized when the argument kx is set to $k\lambda$ with

$$k\lambda = 2\pi \, .$$

The parameter k is the wave number. Remember our graphs for sin(nx)? The “n” told us the number of wave cycles that fit into 2π. Therefore k is literally the wave number.

$$k = \frac{2\pi}{\lambda}$$

In summary, k is indeed the wave number, i.e., how many wavelengths go into 2π. What can we say about kv in sin[k(x – vt)]?

$$\psi_{right}(x,t) = A\sin\left[k(x - vt)\right] = A\sin(kx - kvt)$$

$$\psi_{left}(x,t) = A\sin\left[k(x + vt)\right] = A\sin(kx + kvt)$$

Pick x = 0 so that we are at the origin. Then

$$\psi(t) = \pm\sin(kvt)$$

Includes waves moving left (plus sign) or right (minus sign). One cycle in time is called the period T. A single period is realized when

$$kvT = 2\pi \, .$$

Substituting $k = \dfrac{2\pi}{\lambda}$,

$$kvT = 2\pi \quad \Rightarrow \quad \frac{2\pi}{\lambda} vT = 2\pi \quad \Rightarrow \quad \frac{1}{\lambda} vT = 1$$

$$vT = \lambda \quad \Rightarrow \quad v = \frac{\lambda}{T} \, ,$$

which makes sense since the speed is the time to go one wavelength.

But we just derived our velocity formula from before since frequency $f = \frac{1}{T}$.

Here is our velocity formula or wave relation.

$$v = \frac{\lambda}{T} \quad => \quad v = \lambda f$$

We also have the natural appearance of another important formula from our

$$kvT = 2\pi \, .$$

This formula is equivalent to $kv = \frac{2\pi}{T} = 2\pi f$,

suggesting we define an angular frequency

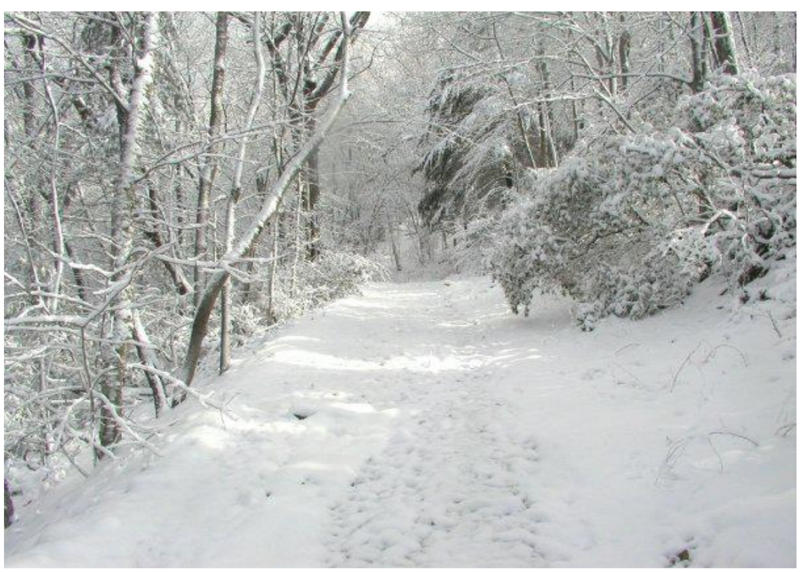
$$\omega = 2\pi f \, .$$

Finally, we can express our rich wave physics with the following.

$$\psi_{right}(x,t) = A\sin(kx - \omega t) \qquad \psi_{left}(x,t) = A\sin(kx + \omega t)$$

$$k = \frac{2\pi}{\lambda} \qquad \omega = 2\pi f \qquad kv = \omega \qquad v = \lambda f$$

I know that you have been introduced to these in a previous course. But remember that I am taking you on a tour through the trail of theoretical physics as it applies to basic optics.



L4. Euler's Formula. Sines and cosines are harder to work with when compared to exponentials. And now we come to a remarkable relation in mathematics.

Leonhard Euler (1707 – 1783)

Let's review your Taylor series

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

for $\sin x$ and $\cos x$. If you do not remember calculating these, I strongly recommend you doing it now. Taking derivatives of cosines and sines are easy. You should be able to derive these results in a very short amount of time.

It will be a good review for you. And you will be acting like a theoretical physicist, where you are powerful and derive all your fundamental physics and mathematics.

For θ in radians, which is really a dimensionless quantity.

$$\cos\theta = 1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-...$$

$$\sin\theta = \theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-...$$

Now stick $i\equiv\sqrt{-1}$ in front of the sine. By the way, i is sacred to electrical engineers as standing for current. So they pick $j\equiv\sqrt{-1}$.

$$i\sin\theta = i\theta-\frac{i\theta^3}{3!}+\frac{i\theta^5}{5!}-...$$

Since $i^2=-1$, $i^3=-i$, $i^4=1$, and $i^5=i$, we can write

$$i\sin\theta = i\theta+\frac{(i\theta)^3}{3!}+\frac{(i\theta)^5}{5!}+...$$

Note that

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots = 1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \ldots$$

Add these together.

$$\cos\theta + i\sin\theta = 1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \ldots + i\theta + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} + \ldots$$

$$\cos\theta + i\sin\theta = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \ldots$$

But wait. Isn't the following the power series for the exponential?

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

Then with $x = i\theta$,

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \ldots$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

This relation is called Euler's formula and Feynman has referred to it as "Our Jewel."

Emerald-Cut Sapphire and Pear-Shaped
Diamond Halo Ring, $50,000
Blue Nile: www.bluenile.com

Now we come to a remarkable identity called Euler's identity. Consider $\theta = \pi$. Then

$e^{i\theta} = \cos\theta + i\sin\theta$ becomes $e^{i\pi} = \cos\pi + i\sin\pi$ and $e^{i\pi} = -1 + 0$, which we can write as follows, the Euler identity.

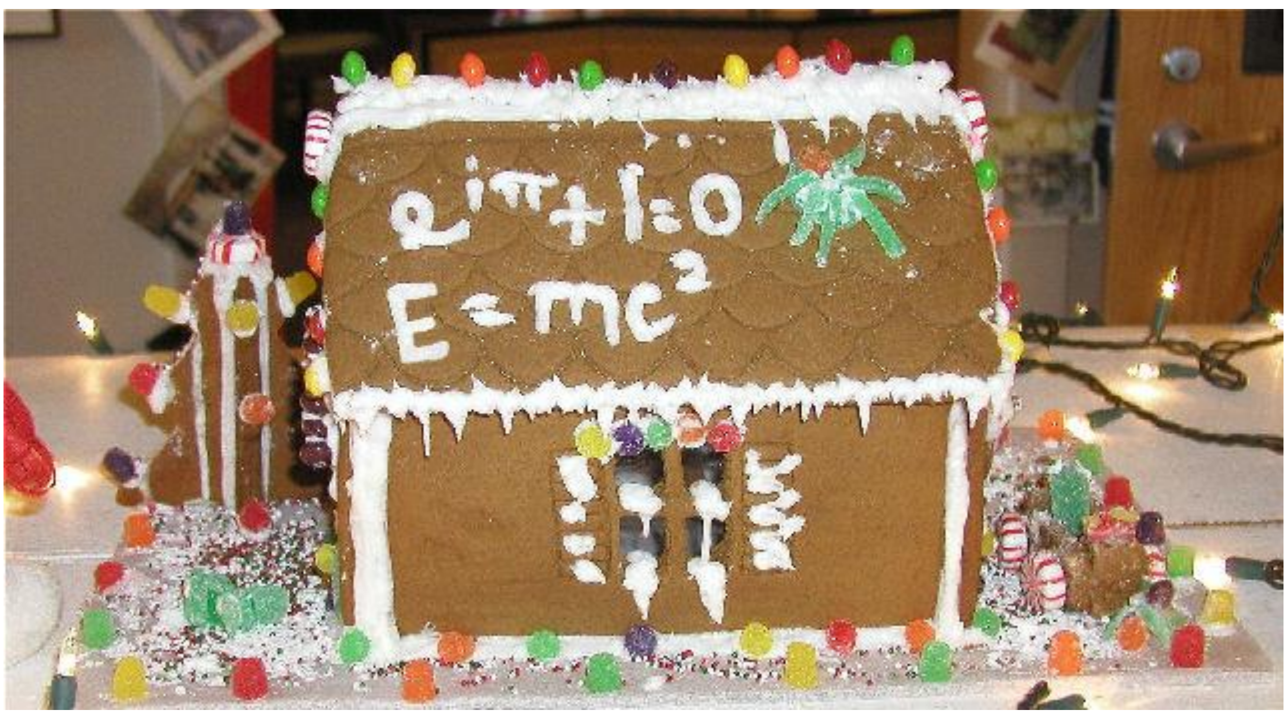
$$\boxed{e^{i\pi} + 1 = 0}$$

I will call this relation "Our Opal."

A 20.05 carat Ethiopian Welo (Wello) opal set in 14k gold and surrounded by diamonds. Wikimedia: Doxymo. Creative Commons

Five most important numbers in mathematics appear here once and only once.

Euler Identity on Rhoades-Robinson Hall Gingerbread House, taking First Prize in the December 2004 Gingerbread Contest.



UNCA Gingerbread Photo by Doc (December Holiday Season 2004)

To illustrate the power of Euler's relation, we proceed to derive some trig identities with ease in the next section. We are in theoretical physics mode now. Derive everything! A fellow grad student from Caltech told me, when we were both at Maryland working on Ph.D. degrees, that Feynman was at the board one day driving integral identities for students, even taking requests!

L5. Euler's Formula and Trig Identities.

$$e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$$

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$

$$= \cos\alpha\cos\beta + i\cos\alpha\sin\beta + i\sin\alpha\cos\beta - \sin\alpha\sin\beta$$

$$= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta)$$

$$\boxed{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

$$\boxed{\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta}$$

$$\sin(\alpha-\beta) = -\cos\alpha\sin\beta + \sin\alpha\cos\beta$$

$$\boxed{\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta}$$

Let $A = \alpha + \beta$ and $B = \alpha - \beta$.

Then $\alpha = \dfrac{A+B}{2}$ and $\beta = \dfrac{A-B}{2}$

$$\boxed{\sin(A) + \sin(B) = 2\sin\left[\frac{A+B}{2}\right]\cos\left[\frac{A-B}{2}\right]}$$

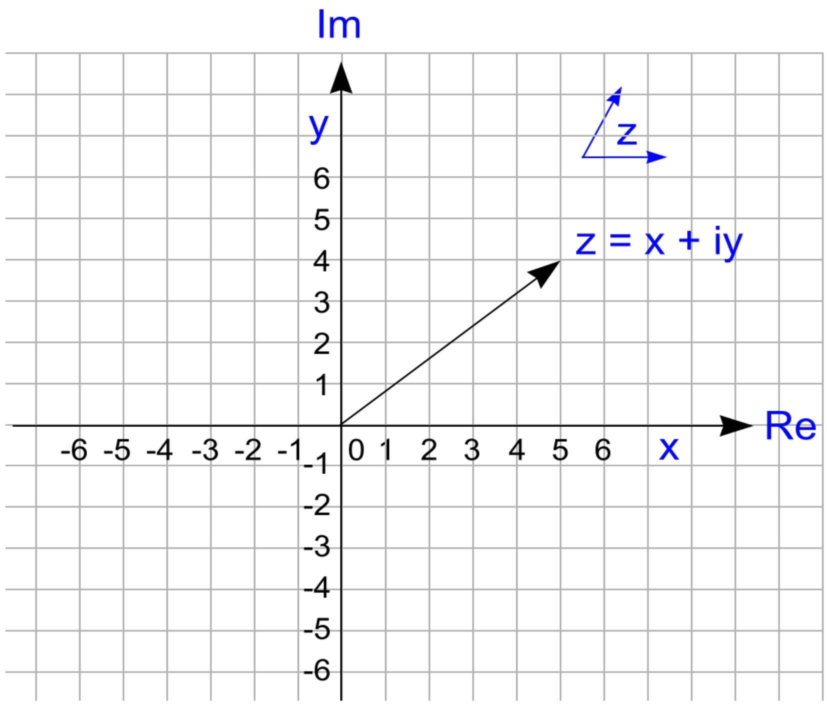
Use $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ with $\alpha = \beta = \theta$.

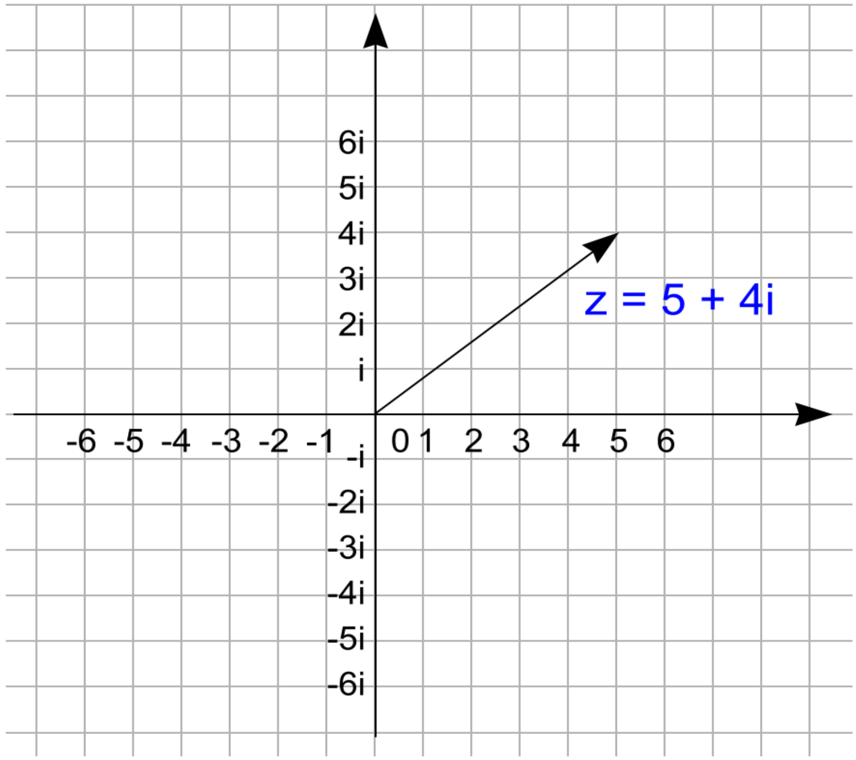
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta \quad \Rightarrow \quad \cos(2\theta) = \cos^2\theta - (1-\cos^2\theta)$$

$$\cos(2\theta) = 2\cos^2\theta - 1$$

$$\frac{1+\cos(2\theta)}{2} = \cos^2\theta \quad \text{and} \quad \boxed{\frac{1+\cos\phi}{2} = \cos^2\frac{\phi}{2}}$$

L6. The Complex Plane and Phasors. Two labelings for the complex plane appear below.





We are going to need some mathematical relationships. The length of the above phasors are given by the Pythagorean theorem.

For the complex number $z = x + iy$, the length of the “vector” in the complex plane is

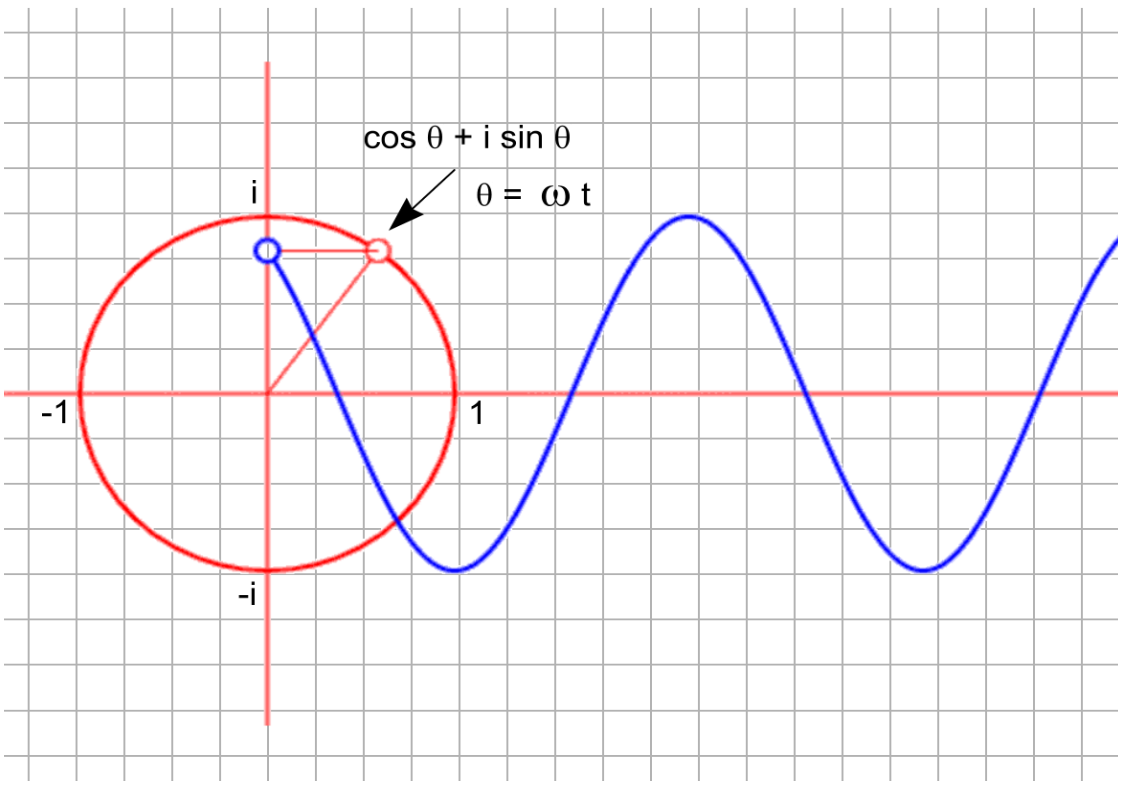
$$|z| = \sqrt{x^2 + y^2} ,$$

which is also called the modulus. The modulus can be thought of as the absolute value of a complex number. The complex conjugate of a complex number $z = x + iy$ is defined as

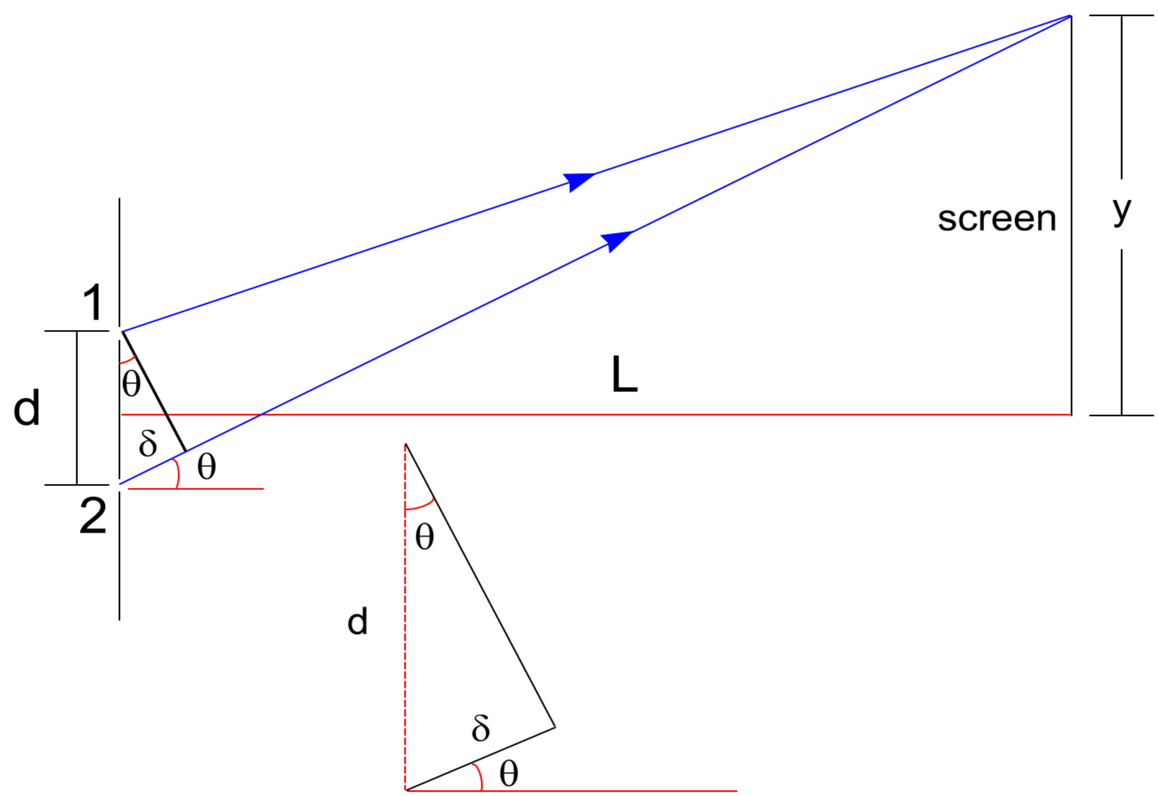
$$z^* = x - iy .$$

You just stick a minus sign whoever there is an i. When we represent a complex number using the Euler formula, many often refer to the number as a **phasor**. Electrical engineers find many, many applications of phasors in circuits with alternative current.

The Phasor $e^{i\theta} = \cos\theta + i\sin\theta$ on the Unit Circle



To illustrate the phasor we consider two light waves interfering similar to what we encountered in an earlier chapter. We apply our wave concepts to the double-slit arrangement shown below.



A light wave, which we will later see is an electromagnetic wave, leaves slits (small openings) 1 and 2. We can write the waves reaching the screen as electric field waves

$$E_1 = E_o \sin(\omega t) \quad \text{and} \quad E_2 = E_o \sin(\omega t + \phi),$$

where the phase ϕ is due to the extra path length δ.

The extra optical path length is

$$\delta = d \sin\theta .$$

A phase of 2π occurs for every wavelength λ of optical path length difference. Therefore,

$$\phi = \frac{\delta}{\lambda} 2\pi .$$

By the way, since $k = \dfrac{2\pi}{\lambda}$, we have this insightful formula $\phi = k\delta$. This relation makes sense since kx appears in $\psi(x,t) = A\sin(kx - \omega t)$ and a delta x translates to a shift in phase: $\phi = k(\Delta x) = k\delta$. For a fixed point on the screen the electric field from slit opening 1 will vary in time according to $E_1 = E_o \sin(\omega t)$. The contribution from slit 2 will include a relative phase: $E_2 = E_o \sin(\omega t + \phi)$. The electric field at the screen is the sum:

$$E = E_1 + E_2 = E_o \sin(\omega t) + E_o \sin(\omega t + \phi).$$

Adding the amplitudes in this fashion is known as the **superposition principle**, i.e., when waves combine, you combine the waves by adding them.

Using our derived identity $\sin(A) + \sin(B) = 2\sin\left[\frac{A+B}{2}\right]\cos\left[\frac{A-B}{2}\right]$, with $A = \omega t + \phi$ and $B = \phi$,

$$E = 2E_o \sin\left[\frac{(\omega t + \phi) + \omega t}{2}\right]\cos\left[\frac{(\omega t + \phi) - \omega t}{2}\right].$$

$$E = 2E_o \sin\left[\frac{2\omega t + \phi}{2}\right]\cos\left[\frac{\phi}{2}\right]$$

$$E = 2E_o \cos\left[\frac{\phi}{2}\right]\sin\left[\omega t + \frac{\phi}{2}\right]$$

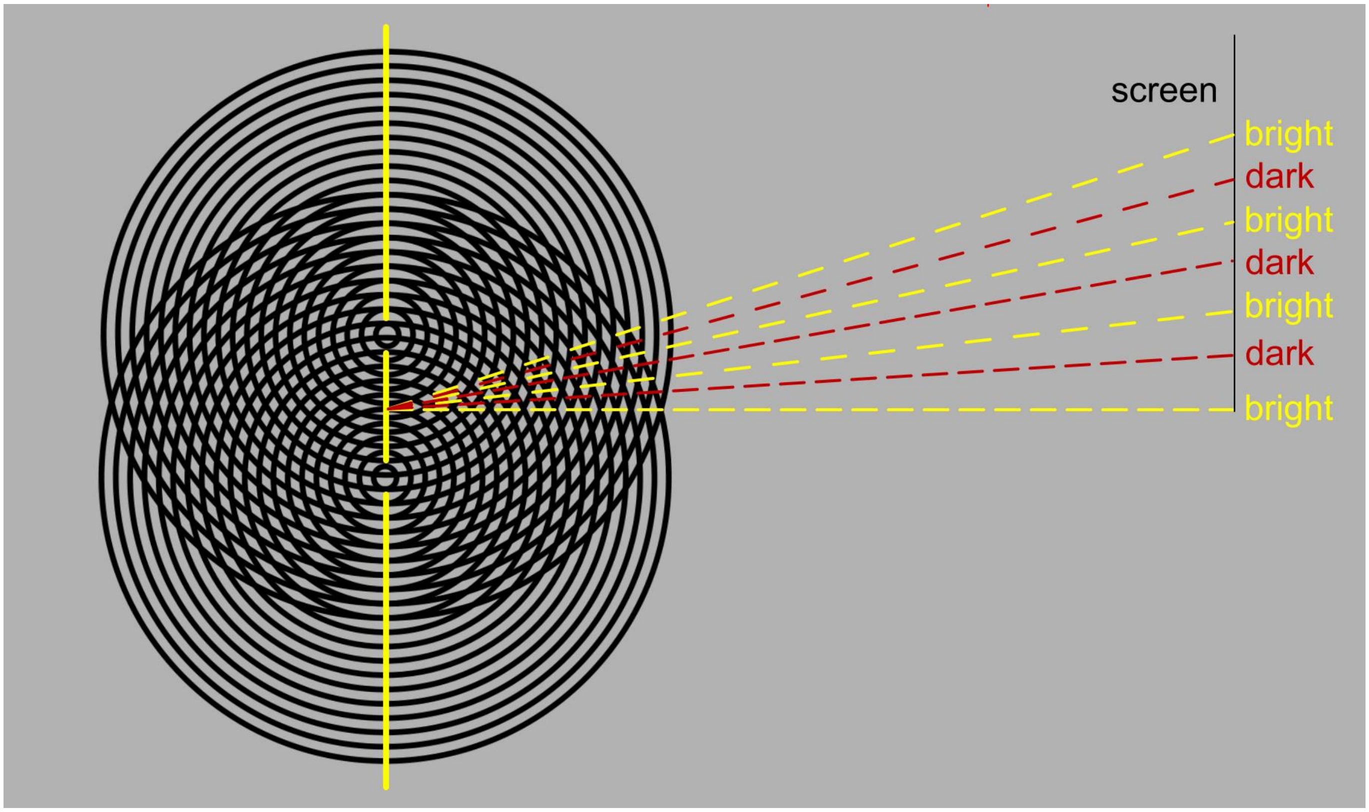
Remember that the potential energy of our harmonic oscillator is $V = \frac{1}{2}kx^2$ and x is a sine wave. Therefore we need to square the amplitude to get the energy. The power is the energy produced per time interval and the intensity is the power per unit area. To simplify all this, just write for the intensity, which will be a measure of brightness,

$$I \sim E^2 = 4E_o^2 \cos^2\left[\frac{\phi}{2}\right]\sin^2\left[\omega t + \frac{\phi}{2}\right]$$

The instantaneous maximum occurs when $\sin^2\left[\omega t + \frac{\phi}{2}\right] = 1$. Therefore,

$$I = I_{\max} \cos^2\left[\frac{\phi}{2}\right] \text{ with } I_{\max} \sim 4E_o^2 .$$

As the optical path difference varies, you will get different levels of brightness as the waves interfere. The max will occur when the waves are in phase at the screen, i.e., crests meet crests and troughs meet troughs. In such a case we have **constructive interference**. When crests meet troughs and vice versa we get darkness, minima. This case is called **destructive interference**. As you move up the screen the optical path difference increases and the phase goes through cycles of constructive and destructive interference. Regions of brightness and darkness will alternate. The overlapping circles below form a display called a Moiré pattern.



Two-Slit Interference using Moirè Circles taken from Wikimedia: SharkD. Creative Commons

Remember the importance of doing calculations in more than one way? What about phasors?

$$E = E_1 + E_2 = E_o \sin(\omega t) + E_o \sin(\omega t + \phi) \rightarrow E_o e^{i\omega t} + E_o e^{i(\omega t + \phi)}$$

$$E = E_o e^{i\omega t}\left(1 + e^{i\phi}\right)$$

$$I \sim |E|^2 = \left[E_o e^{i\omega t}\left(1 + e^{i\phi}\right)\right]\left[E_o e^{i\omega t}\left(1 + e^{i\phi}\right)\right]^*$$

$$I \sim E_o e^{i\omega t}\left(1 + e^{i\phi}\right) E_o e^{-i\omega t}\left(1 + e^{-i\phi}\right)$$

$$I \sim E_o^2\left(1+e^{i\phi}\right)\left(1+e^{-i\phi}\right) = E_o^2\left(1+e^{i\phi}+e^{-i\phi}+1\right)$$

$$I \sim E_o^2\left(2+e^{i\phi}+e^{-i\phi}\right)$$

$$e^{i\phi}+e^{-i\phi} = \left(\cos\phi + i\sin\phi\right)+\left(\cos\phi - i\sin\phi\right) = 2\cos\phi$$

$$I \sim E_o^2\left(2+2\cos\phi\right) = 2E_o^2(1+\cos\phi)$$

$$\boxed{\frac{1+\cos\phi}{2} = \cos^2\frac{\phi}{2}}$$

$$I \sim 4E_o^2\cos^2\left[\frac{\phi}{2}\right]$$

$I = I_{\max}\cos^2\left[\dfrac{\phi}{2}\right]$ with $I_{\max} \sim 4E_o^2$, as we found before.

Note that $e^{i\theta} = \cos\theta + i\sin\theta$ with $e^{-i\theta} = \cos\theta - i\sin\theta$ leads to

$$e^{i\theta}+e^{-i\theta} = 2\cos\theta \quad \text{and} \quad e^{i\theta}-e^{-i\theta} = 2\sin\theta$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

Very powerful stuff.

L7. Beats. Beats occur when two waves of equal amplitude and nearly the same frequency are added. The waves drift in and out of phases with a pulsating effect. The variation in strength is called an **amplitude modulation**. We choose phases to be zero at x = 0 and t = 0 for the two waves.

$$E_1 = E_o\sin(k_1x-\omega_1t) \qquad E_2 = E_o\sin(k_2x-\omega_2t)$$

$$E = E_1+E_2 = E_o\sin(k_1x-\omega_1t)+E_o\sin(k_2x-\omega_2t)$$

Let $A = k_1 x - \omega_1 t$ and $B = k_2 x - \omega_2 t$. Use the following identity we derived earlier.

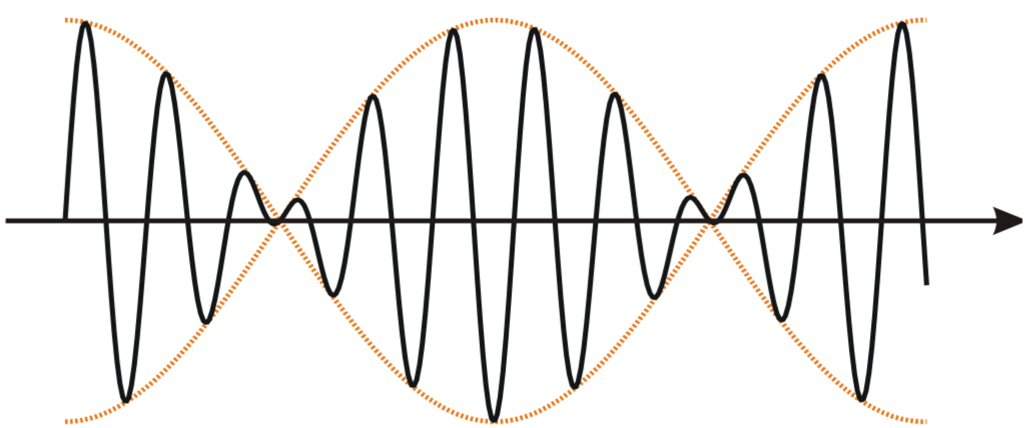
$$\boxed{\sin(A) + \sin(B) = 2\sin\left[\frac{A+B}{2}\right]\cos\left[\frac{A-B}{2}\right]}$$

$$E = 2E_o \sin\left[\frac{(k_1+k_2)x}{2} - \frac{(\omega_1+\omega_2)t}{2}\right]\cos\left[\frac{(k_1-k_2)}{2} - \frac{(\omega_1-\omega_2)t}{2}\right]$$

Let $\bar{k} = \dfrac{k_1+k_2}{2}$, $\bar{\omega} = \dfrac{\omega_1+\omega_2}{2}$, $\Delta k = k_1 - k_2$, and $\Delta\omega = \omega_1 - \omega_2$.

$$E = 2E_o \sin\left[\bar{k}x - \bar{\omega}t\right]\cos\left[\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right]$$

$$E = 2E_o \cos\left[\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right]\sin\left[\bar{k}x - \bar{\omega}t\right]$$



Wikipedia: Ansgar Hellwig. Creative Commons

The beat frequency is $2\Delta\bar{\omega}$. Why?

The $\sin\left[\bar{k}x - \bar{\omega}t\right] = \sin\left[\bar{k}(x - \dfrac{\bar{\omega}}{\bar{k}})t\right]$ wave is called the **carrier wave** with wave velocity

$$v_p = \frac{\bar{\omega}}{\bar{k}}$$, named the **phase velocity**.

The envelope modulation factor $\cos\frac{1}{2}[\Delta k x - \Delta\omega t] = \cos\frac{1}{2}\left[\Delta k(x - \frac{\Delta\omega}{\Delta k}t)\right]$ advances

at the speed $v_g = \frac{\Delta\omega}{\Delta k}$, which is called the **group velocity**.

Here $v_1 = \frac{\omega_1}{k_1}$ and $v_2 = \frac{\omega_2}{k_2}$ will be equal if the speed does not depend on wavelength.

$$v = \frac{\omega_1}{k_1} = \frac{\omega_2}{k_2}$$

Then $v_p = \frac{\bar{\omega}}{\bar{k}} = \frac{(\omega_1 + \omega_2)/2}{(k_1 + k_2)/2} = \frac{\omega_1 + \omega_2}{k_1 + k_2} = \frac{\frac{\omega_1}{k_1} + \frac{\omega_2}{k_1}}{1 + \frac{k_2}{k_1}} = \frac{v + \frac{\omega_2}{k_1}}{1 + \frac{k_2}{k_1}}$.

Now use $v_2 = \frac{\omega_2}{k_2} = v$ in the form of $\omega_2 = k_2 v$.

$$v_p = \frac{v + \frac{\omega_2}{k_1}}{1 + \frac{k_2}{k_1}} = \frac{v + \frac{k_2 v}{k_1}}{1 + \frac{k_2}{k_1}} = \frac{v(1 + \frac{k_2}{k_1})}{1 + \frac{k_2}{k_1}} = v$$.

Likewise $v_g = \frac{\Delta\omega}{\Delta k} = \frac{\omega_1 - \omega_2}{k_1 - k_2} = \frac{\frac{\omega_1}{k_1} - \frac{\omega_2}{k_1}}{1 - \frac{k_2}{k_1}} = \frac{v - \frac{k_2 v}{k_1}}{1 - \frac{k_2}{k_1}} = \frac{v(1 - \frac{k_2}{k_1})}{1 - \frac{k_2}{k_1}} = v$.

$$v_g = v_p = v_1 = v_2 = v$$

Nondispersive Medium: “Everybody” travels at the same speed!

L8. Group Velocity and Dispersion. If the wave travels in a medium other than vacuum, there will be dispersion. We now relate this chapter (L) the previous chapter (J). In general

$$v_g = \frac{d\omega}{dk} \neq v_p \,.$$

Start with $\omega = kv$ and take the derivative using the product rule.

$$v_g = \frac{d\omega}{dk} = v + k\frac{dv}{dk} \,.$$

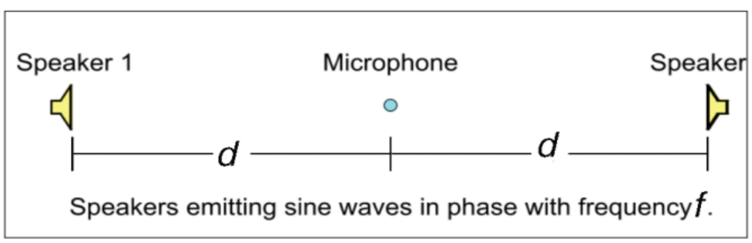
Now use $n = \frac{c}{v}$ in the form $v = \frac{c}{n}$.

$$v_g = \frac{d\omega}{dk} = \frac{c}{n} + k\frac{d}{dk}\left[\frac{c}{n}\right] = \frac{c}{n} + k\left[-\frac{c}{n^2}\right]\frac{dn}{dk}$$

$$v_g = \frac{c}{n}\left[1 - \frac{k}{n}\frac{dn}{dk}\right]$$

L9. A Treacherous Trap. "To illustrate the treacherous nature of the convention described above for waves traveling to the right and left," see the "arrangement where two speakers are separated by a distance 2d. The speakers emit sine waves in phase, which are picked up by the microphone at the middle position between the two speakers."

James Perkins and Michael J. Ruiz, "A Reliable Wave Convention for Oppositely Traveling Waves," *The Physics Teacher* **56**, 622 (December 2018). **pdf**



"Let the wave leaving the left speaker be

$$y_R(x,t) = A \sin (kx - \omega t),$$

where the subscript R designates that the wave travels to the right. For the wave emitted from the right speaker, we arrive at the function by shifting a sine wave a distance 2d to the right and changing $-\omega t$ to $+\omega t$ so that the wave moves to the left:

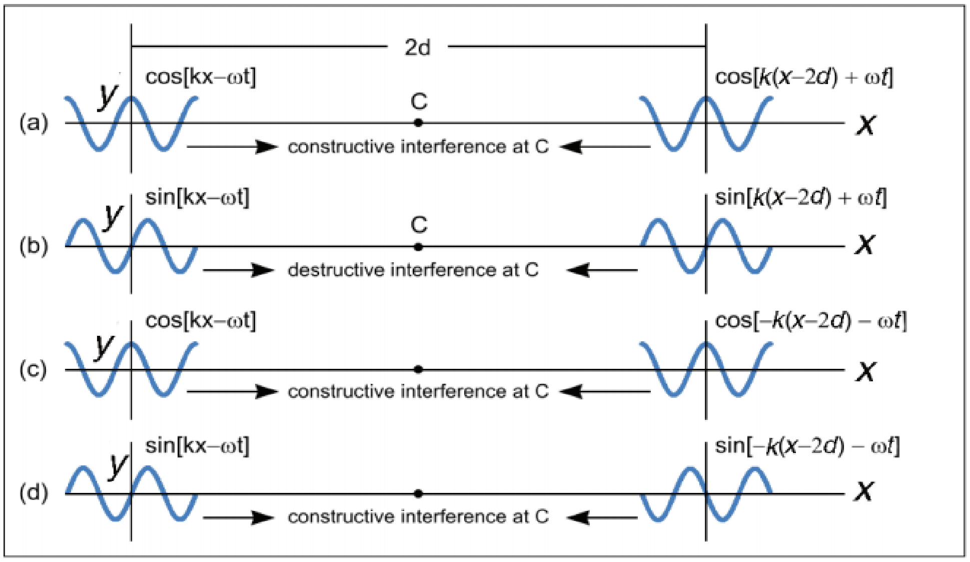
$$y_L(x,t) = A \sin[k(x - 2d) + \omega t].$$

The superposition of the waves at the microphone, i.e., at x = d, is then

$$y(d,t) = A \sin (kd - \omega t) + A \sin(-kd + \omega t) = 0,$$

which disagrees with the experimental observation." (Perkins and Ruiz, 2018)

The figure below will help us see our way through this paradox. Our convention is introducing a subtle 180° phase shift at the right for the sine waves! This phase shift creeping in unbeknownst to us leads to the paradox – our erroneous conclusion that destructive interference occurred at the center of our two-speaker arrangement. The phase shift does not enter when you switch the minus sign in front of the kx instead of the ωt for the left-traveling wave



"Cosine and sine waves are first shifted a distance 2d to the right, cosine waves for (a) and (c); sine waves for (b) and (d). The convention where the sign of ω changes is used for the left-traveling waves in (a) and (b); the convention where the sign of k changes is employed for the left-traveling waves in (c) and (d). The latter convention gives the correct conclusion that constructive interference occurs at the center point for both the cosine (c) and sine (d) waves."

So it is best to use

$$\psi_{right}(x,t) = A\sin(kx - \omega t) \qquad \psi_{left}(x,t) = A\sin(-kx - \omega t)$$

$$\psi_{right}(x,t) = A\cos(kx - \omega t) \qquad \psi_{left}(x,t) = A\cos(-kx - \omega t)$$

$$\psi_{right}(x,t) = Ae^{i(kx-\omega t)} \qquad \psi_{left}(x,t) = Ae^{-i(kx+\omega t)}$$

L10. Summing Multiple Waves. We can consider a superposition of many waves by an integral. Here is a more general wave traveling to the right.

$$\psi(x,t) = \int A(k)e^{i(kx-\omega t)}dk$$

When the wave is localized, we call it a **wave packet**.