Q0. Wave DE (Differential Equation).

Earlier we found in one dimension.

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

The two-dimensional differential wave equation is

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}.$$

The three-dimensional differential wave equation is

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}.$$

Q1. Operator Notation.

Del operator

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\nabla\cdot\nabla = (\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z})\cdot(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z})$$

$$\nabla\cdot\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Pierre-Simon Laplace (1749 – 1827)
Known in physics/math/engineering especially for the Laplacian and the Laplace Transform

Due to his superb intellect and accomplishments, he became known as the “Newton of France.”

The Laplacian is defined below.

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

We can now write the three-dimensional wave equation in a compact form.

$$\nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} .$$

Replacing the general speed v with c, the speed of light, we get the wave equation in vacuum.

$$\boxed{\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}}$$

We can also write

$$\frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi = 0$$

$$\frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial z^2} = 0$$

Whenever space and time appear like $c^2 t^2 - x^2 = 0$ or $c^2 t^2 - x^2 - y^2 - z^2 = 0$,
The equation is consistent with Einstein’s Theory of Relativity.
We say the equation is relativistically invariant.

The combination $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2})$ qualifies.

When Einstein comes along, Einstein refers to the variables as four dimensions.
The time dimension must be added in the form of ct or c dt so the dimensions agree.

Jean le Rond d'Alembert (1717 – 1783)
Mathematician, Physicist, Music Theorist

Known to the general population as the co-editor with Denis Diderot of the first encyclopedia.

Discoverer of the Wave Equation in One Dimension

$$\frac{\partial^2\psi}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}.$$

The general solution we gave earlier to the one-dimensional wave equation is also due to d'Alembert.

$$\psi(x,t)=f(x-vt)+g(x+vt)$$

The d'Alembert operator, commonly known as the d'Alembertian, is named after him. It is also called the box operator:

$$\Box\equiv\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}.$$

Now the wave equation in 3D (three spatial dimensions are implied) is super concise!

$$\boxed{\Box\psi=0}$$

Q2. Coordinate Systems. Everything is much simpler in Cartesian coordinates.

René Descartes (1596 – 1650)
Philosopher, Mathematician, Physicist

Known for philosophy: “I think, therefore I am.”
Often called the “father of modern philosophy.”

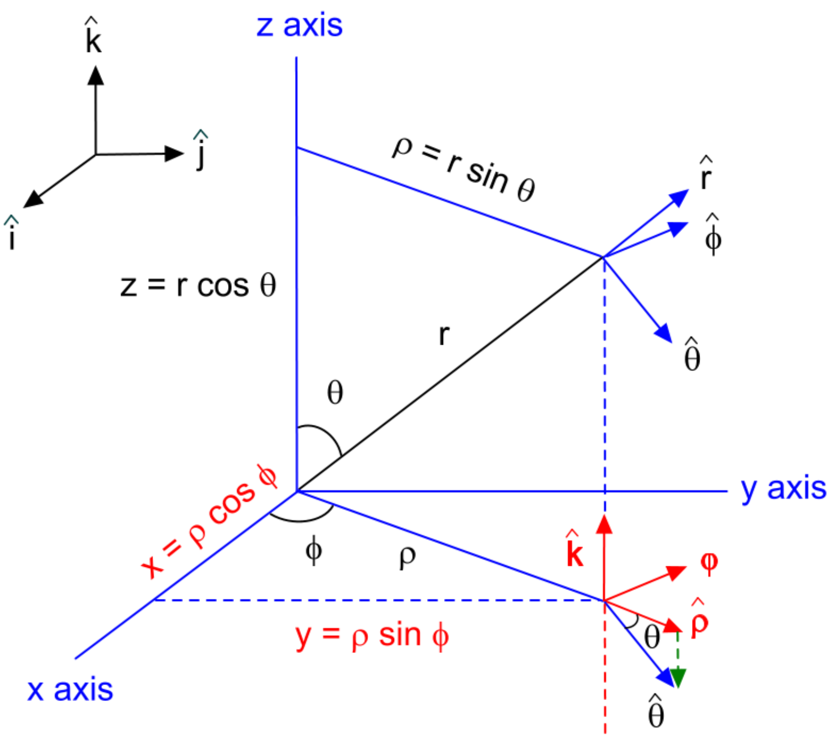
The Cartesian coordinate system (x,y).

Developed Analytic Geometry.

Gave a Derivation of the Law of Refraction.

Calculated the 42° for the Primary Rainbow before the advent of calculus!

The Laplacian takes on its simplest form in Cartesian coordinates: $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.



a) Cartesian Coordinates (x,y,z).

When z = 0, you get the very popular (x,y) plane system.

b) Cylindrical Coordinates (ρ,φ,z)

When z = 0, you get polar coordinates: (ρ,φ).

c) Spherical Coordinates (r,φ,θ).

All three coordinate systems are shown in the left figure. Note that each coordinate system has its unit vectors.

Very Important Observation: Unit vectors for r, φ, θ, and ρ are NOT constant!

With various coordinate systems, the Laplacian is often used in physics, chemistry, and engineering.

Mechanics – Poisson's Equation:

$\nabla^2\phi = 4\pi G\rho$, where ϕ is potential, ρ mass density.

Electromagnetic Theory – Poisson's Equation in Electrostatics:

$\nabla^2\phi = -\dfrac{\rho}{\varepsilon_o}$, where ϕ is potential, ρ charge density.

Thermodynamics – the Heat Equation:

$\dfrac{\partial T}{\partial t} = \alpha\nabla^2 T$, where T is temperature.

Quantum Mechanics – the time-independent Schrödinger Equation:

$$-\frac{h^2}{2m}\nabla^2\psi + V\psi = E\psi$$

Optics – the Wave Equation:

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

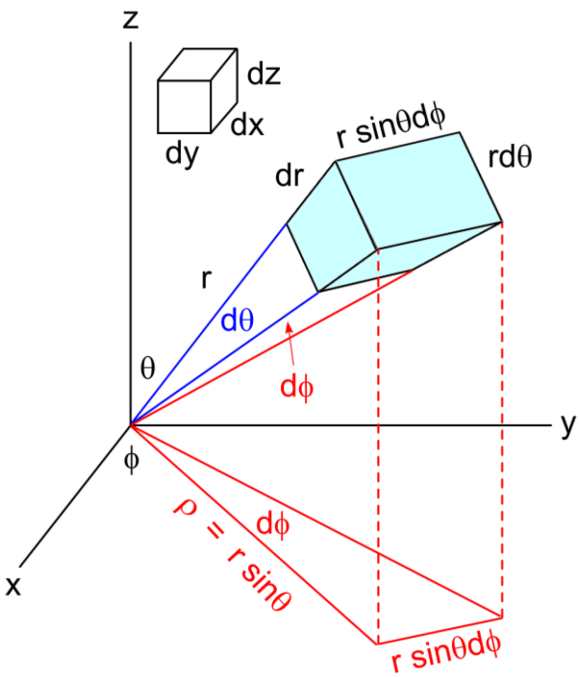
Have you ever encountered the Laplacian in spherical coordinates?

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Yikes!

To understand the above operator in spherical coordinates, we turn to curvilinear coordinates.

Q4. Curvilinear Coordinates. Remember your volume elements in Cartesian, cylindrical, and spherical coordinate systems? The easy one to remember is the (dx)(dy)(dz) of Cartesian coordinates. Below I show you how to get the spherical and cylindrical cases.



Cartesian: $dV = dxdydz$

Cylindrical: $dV = \rho d\rho d\phi dz$

Spherical: $dV = r^2 \sin\theta dr d\theta d\phi$

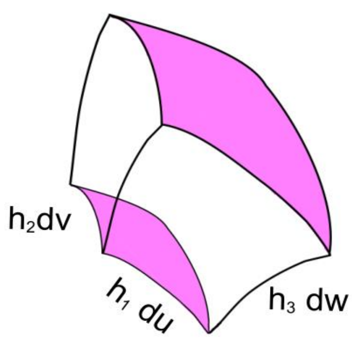
Refer to the left figure to get the spherical volume element of the cute cube-like blue volume.

$$dV = (dr)(rd\theta)(r\sin\theta d\phi)$$

$$dV = r^2 \sin\theta dr d\theta d\phi$$

To get the cylindrical case, increase θ to 90° so that you are lowering the volume element in the figure by swinging it down. In the x-y plane

$(dr)(r\sin\theta)d\phi \to (d\rho)(\rho d\phi)$ and $rd\theta \to dz$ to arrive at $dV = \rho d\rho d\phi dz$. But it is simpler to think polar coordinates first and write $(d\rho)(\rho d\phi)$ in the x-y plane for the polar coordinates. Then, tack on dz for the third dimension.



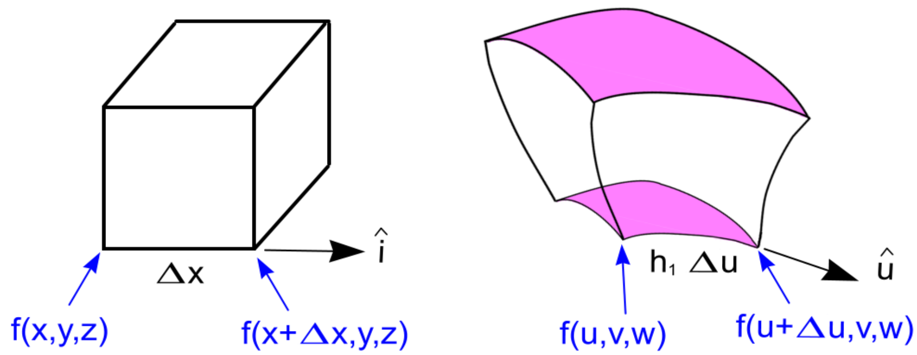
All three of our coordinate systems have volume elements of the form

$$dV = (h_1 du)(h_2 dv)(h_3 dw).$$

u	v	w	h_1	h_2	h_3
x	y	z	1	1	1
ρ	ϕ	z	1	ρ	1
r	θ	ϕ	1	r	r sinθ

The general cases of such coordinates are called **curvilinear coordinates**. The volume element in the upper figure on this page really has "curvy" lines for several of the little delta lengths. The unit vectors are $\hat{u}$, $\hat{v}$, and $\hat{w}$. Lots of books use the notation for the coordinates as x_1, x_2, and x_3 or q_1, q_2, and q_3. Unit vectors are also commonly $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$.

Q5. The Gradient. We will build up to $\nabla^2\psi$ eventually. First consider something simpler, the gradient.



Derivative: $$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Partial derivative: $$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x,y,z)}{\Delta x}$$

Directional derivative: $$\text{grad } f \equiv \left[\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \right] \hat{i} = \frac{df}{dx} \hat{i}$$

Directional partial derivative: $$\text{grad } f \equiv \left[\lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x,y,z)}{\Delta x} \right] \hat{i} = \frac{\partial f}{\partial x} \hat{i}$$

Curvilinear version: $$\text{grad } f \equiv \left[\lim_{\Delta u \to 0} \frac{f(u+\Delta u, v, w) - f(u,v,w)}{h_1 \Delta u} \right] \hat{i} = \frac{1}{h_1} \frac{\partial f}{\partial u} \hat{u}$$

Cartesian coordinates: $$\text{grad } f = \nabla f = \hat{i} \frac{\partial f}{\partial x} + \hat{j} \frac{\partial f}{\partial y} + \hat{k} \frac{\partial f}{\partial z}.$$

In general: $$\boxed{\nabla f = \hat{e}_1 \frac{1}{h_1} \frac{\partial f}{\partial q_1} + \hat{e}_1 \frac{1}{h_1} \frac{\partial f}{\partial q_2} + \hat{e}_1 \frac{1}{h_1} \frac{\partial f}{\partial q_3}}$$

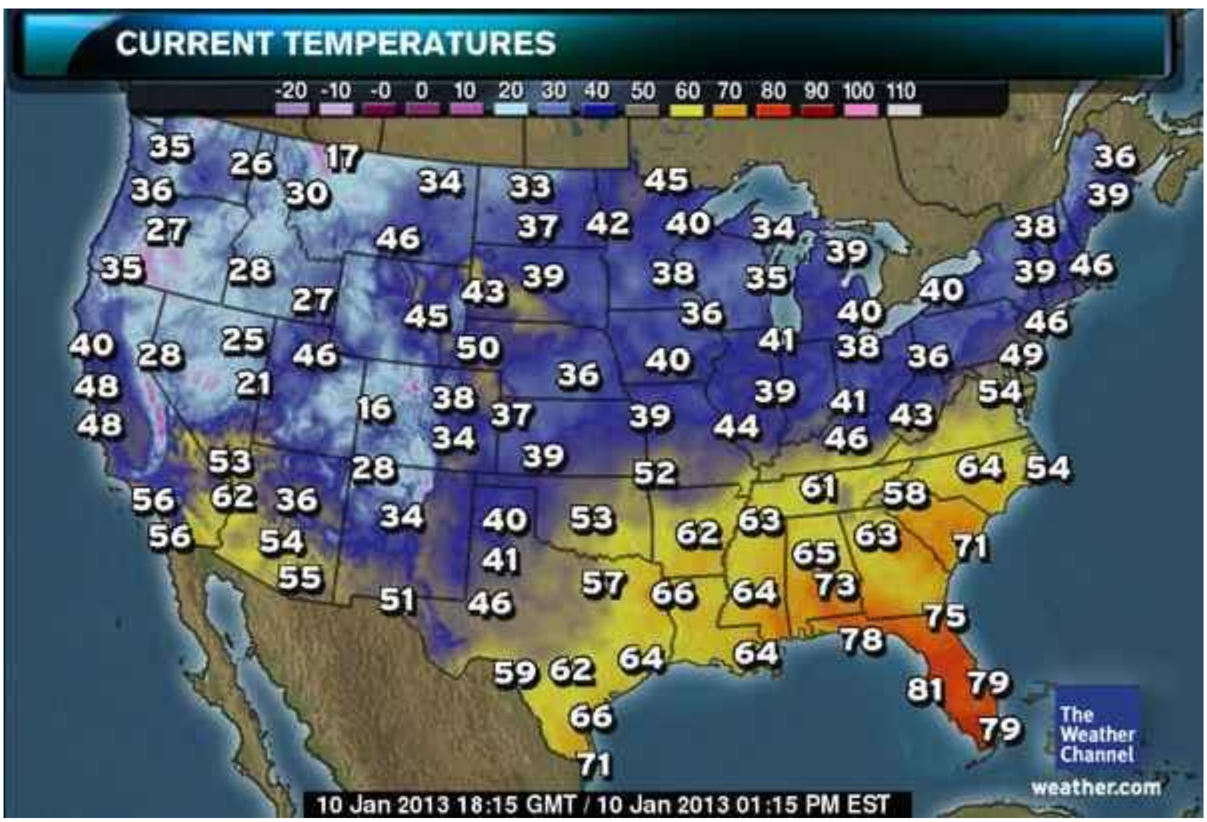
Gradient Summary

Cartesian: $$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

Cylindrical: $$\nabla f = \hat{\rho}\frac{\partial f}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial f}{\partial \phi} + \hat{k}\frac{\partial f}{\partial z}$$

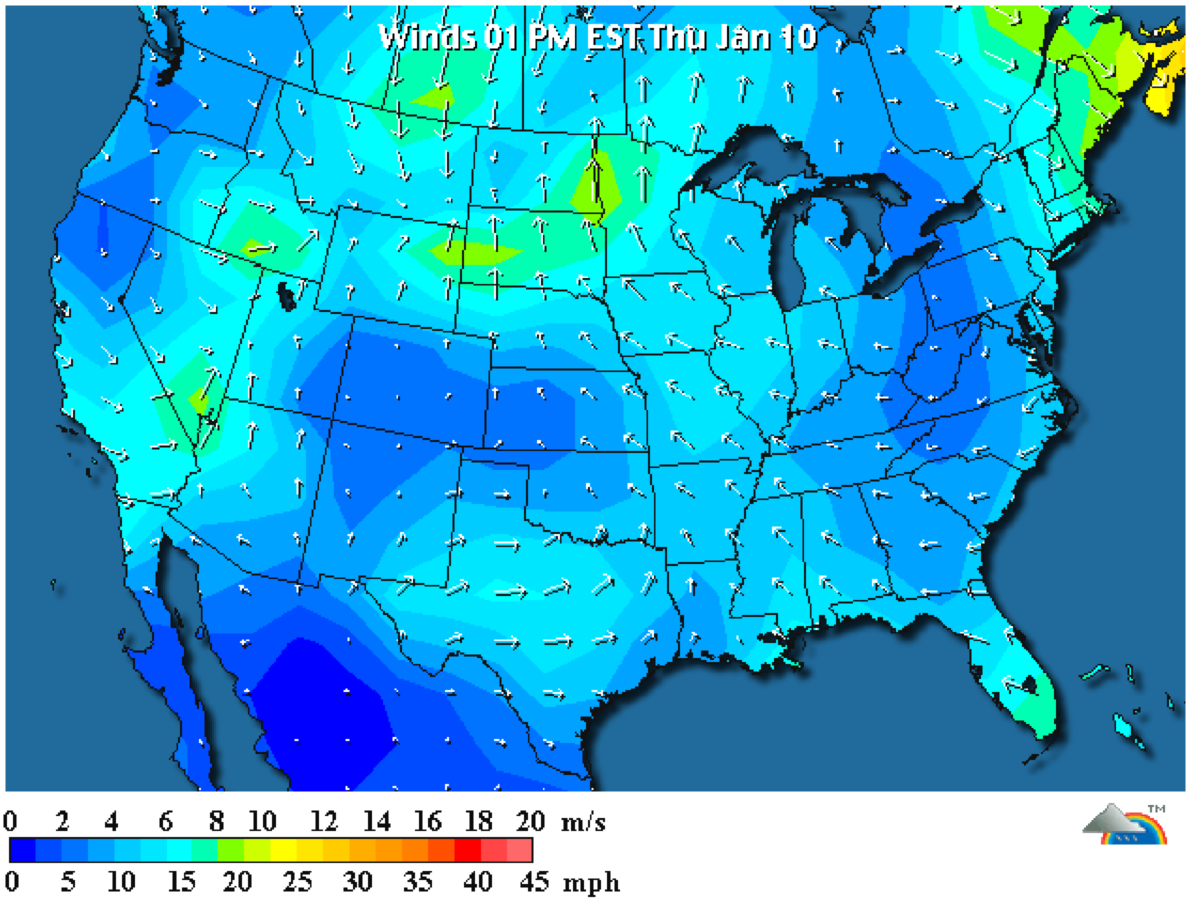
Spherical: $$\nabla f = \hat{r}\frac{\partial f}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

We have promoted the function f to a vector. When we have something like temperature with its single magnitude value such as 20° C, we call it a **scalar**. See the map below with temperatures.



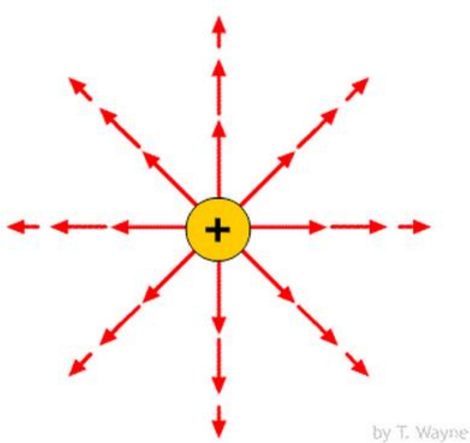
Courtesy The Weather Channel

If we have an entity with magnitude and direction, we call it a **vector**. The gradient promotes a scalar to a vector. See the next figure for an example of a gradient plot. The gradient plot gives little vector arrows. The length of the vector indicates its strength or magnitude and the little arrow gives the direction. Wind speed and direction together make up a vector quantity.



Courtesy Weather Underground, Inc.

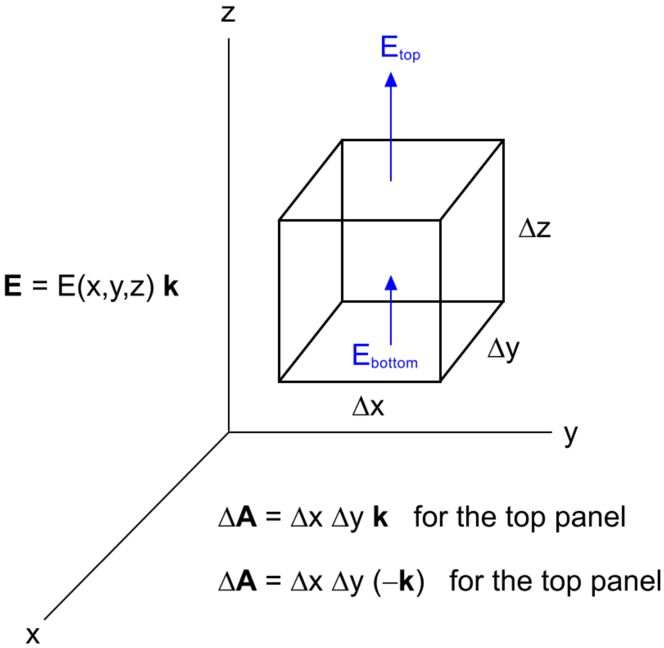
Wind velocity has magnitude (the speed) and direction. The length of the vector arrows indicate the magnitude of the velocity and the arrow points in the direction of the wind. Technically, speed is a scalar, the magnitude. When you promote speed to a vector you add the direction. However, often velocity is used informally for just speed.



Charge Image Courtesy Tony Wayne

Here is a vector field produced by a plus charge. Note the symmetry as all vectors points outward away from the positive charge. Also note that the lengths of the vectors decrease as you get farther away from the charge. The strength weakens according to the inverse square law. In contrast to the weather case, this field has a simple formula.

Q6. The Divergence. Here is a simplified derivation of the divergence theorem in Cartesian coordinates.



We are interested in calculating the flux through the enclosed surface, which we write as

$$\oint\!\!\oint \vec{E} \cdot \vec{dA}.$$

We have $E_{\text{bottom}} = E_z(x, y, z)$ and its counterpart $E_{\text{top}} = E_z(x, y, z + \Delta z)$.

The net flux out of the surface of our cube is given by multiplying the magnitude of the perpendicular vector component that pierces each surface. Here we have top and bottom. We subtract what goes in from what goes out.

$$\oint\!\!\oint \vec{E} \cdot \vec{dA} => E_z(x, y, z + \Delta z)\Delta x \Delta y - E_z(x, y, z)\Delta x \Delta y$$

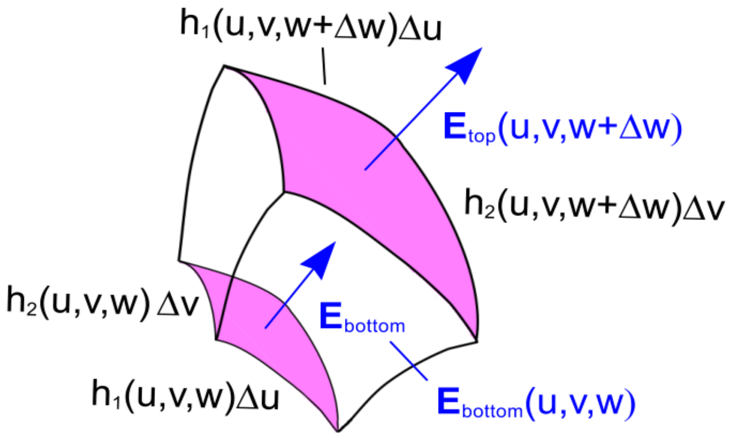
$$\oint\!\!\oint \vec{E} \cdot \vec{dA} => \frac{E_z(x, y, z + \Delta z) - E_z(x, y, z)}{\Delta z} \Delta x \Delta y \Delta z$$

$$\oint\!\!\oint \vec{E} \cdot \vec{dA} = \iiint \frac{\partial E_z}{\partial z} dxdydz$$ Note left surface integral and right volume integral.

$$\oint\!\!\oint \vec{E} \cdot \vec{dA} = \iiint \left[\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right] dxdydz$$

The Divergence Theorem: $$\oint\!\!\oint \vec{E} \cdot \vec{dA} = \iiint \left[\nabla \cdot \vec{E} \right] dV.$$

The divergence in Cartesian coordinates is $$\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}.$$



$$\oiint \vec{E} \cdot \overrightarrow{dA}.$$

.

We have $E_{\text{bottom}} = E_w(u, v, w)$ and its counterpart

$$E_{\text{top}} = E_w(u, v, w + \Delta w).$$

The net flux out of the surface of our small enclosure is given by multiplying the magnitude of the perpendicular vector component that pierces each surface. Here we have top and bottom. We subtract what goes in from what goes out.

$$\oiint \vec{E} \cdot \overrightarrow{dA} => \left[E_w (h_1 \Delta u)(h_2 \Delta v)\right]_{(u,v,w+\Delta w)} - \left[E_w (h_1 \Delta u)(h_2 \Delta v)\right]_{(u,v,w)}$$

$$\oiint \vec{E} \cdot \overrightarrow{dA} => \left[E_w h_1 h_2 \Big|_{(u,v,w+\Delta w)} - E_w h_1 h_2 \Big|_{(u,v,w)} \right] \Delta u \Delta v$$

$$\oiint \vec{E} \cdot \overrightarrow{dA} => \Delta(E_w h_1 h_2) \Delta u \Delta v = \frac{\Delta(E_w h_1 h_2)}{\Delta w} \Delta u \Delta v \Delta w$$

$$\oiint \vec{E} \cdot \overrightarrow{dA} => \frac{\partial(E_w h_1 h_2)}{\partial w} \Delta u \Delta v \Delta w$$

$$\oiint \vec{E} \cdot \overrightarrow{dA} = \iiint \frac{\partial(E_w h_1 h_2)}{\partial w} dudvdw$$

$$dV = (h_1 du)(h_2 dv)(h_3 dw) \quad => \quad \oiint \vec{E} \cdot \overrightarrow{dA} = \iiint \frac{\partial(E_w h_1 h_2)}{\partial w} \frac{1}{h_1 h_2 h_3} dV$$

$$\oiint \vec{E} \cdot \overrightarrow{dA} = \iiint \left[\frac{\partial(E_u h_2 h_3)}{\partial u} + \frac{\partial(E_v h_1 h_3)}{\partial v} + \frac{\partial(E_w h_1 h_2)}{\partial w} \right] \frac{1}{h_1 h_2 h_3} dV$$

The divergence in curvilinear coordinates is

$$\nabla \cdot \vec{E} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(E_u h_2 h_3)}{\partial u} + \frac{\partial(E_v h_1 h_3)}{\partial v} + \frac{\partial(E_w h_1 h_2)}{\partial w}\right].$$

$$\boxed{\nabla \cdot \vec{E} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(E_1 h_2 h_3)}{\partial q_1} + \frac{\partial(E_2 h_1 h_3)}{\partial q_2} + \frac{\partial(E_3 h_1 h_2)}{\partial q_3}\right]}$$

Cartesian: $(q_1, q_2, q_3) = (x, y, z)$ and $(h_1, h_2, h_3) = (1,1,1)$.

$$\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

Cylindrical: $(q_1, q_2, q_3) = (\rho, \phi, z)$ and $(h_1, h_2, h_3) = (1, \rho, 1)$.

$$\nabla \cdot \vec{E} = \frac{1}{1 \cdot \rho \cdot 1}\left[\frac{\partial(E_\rho \rho \cdot 1)}{\partial \rho} + \frac{\partial(E_\phi \cdot 1 \cdot 1)}{\partial \phi} + \frac{\partial(E_z \cdot 1 \cdot \rho)}{\partial z}\right]$$

$$\nabla \cdot \vec{E} = \frac{1}{\rho}\frac{\partial(\rho E_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z}$$

Spherical: $(q_1, q_2, q_3) = (r, \theta, \phi)$ and $(h_1, h_2, h_3) = (1, r, r\sin\theta)$.

$$\nabla \cdot \vec{E} = \frac{1}{r^2 \sin\theta}\left[\frac{\partial(E_r r \cdot r\sin\theta)}{\partial r} + \frac{\partial(E_\theta \cdot 1 \cdot r\sin\theta)}{\partial \theta} + \frac{\partial(E_\phi \cdot 1 \cdot r)}{\partial \phi}\right]$$

$$\nabla \cdot \vec{E} = \frac{1}{r^2}\frac{\partial(r^2 E_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta E_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial \phi}$$

Q7. The Laplacian. Recall earlier that we defined the Laplacian as

$$\nabla \cdot \nabla = (\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}) \cdot (\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} .$$

It is instructive to consider this operator as actually operating on something: $\nabla^2 f$.

We see that the Laplacian of a function, i.e., $\nabla^2 f$, is equivalent to applying the divergence to the gradient of a function: $\nabla^2 f = \nabla \cdot (\nabla f)$. We start with our previous general result for the gradient

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial q_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial q_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\hat{e}_3 .,$$

and then use our previous result for the divergence

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(A_1 h_2 h_3)}{\partial q_1} + \frac{\partial(A_2 h_1 h_3)}{\partial q_2} + \frac{\partial(A_3 h_1 h_2)}{\partial q_3}\right],$$

where $\vec{A} = \nabla f$. So we substitute into $\nabla \cdot \vec{A}$ our components of $\vec{A} = \nabla f$, which are

$$A_1 = \frac{1}{h_1}\frac{\partial f}{\partial q_1}, \quad A_2 = \frac{1}{h_2}\frac{\partial f}{\partial q_2}, \text{ and } A_3 = \frac{1}{h_3}\frac{\partial f}{\partial q_3} .$$

Then we obtain for $\nabla^2 f$ the following.

$$\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(\frac{1}{h_1}\frac{\partial f}{\partial q_1} h_2 h_3) + \frac{\partial}{\partial q_2}(\frac{1}{h_2}\frac{\partial f}{\partial q_2} h_1 h_3) + \frac{\partial}{\partial q_3}(\frac{1}{h_3}\frac{\partial f}{\partial q_3} h_1 h_2)\right]$$

This expression simplifies to

$$\boxed{\nabla^2 = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(\frac{h_2 h_3}{h_1}\frac{\partial}{\partial q_1}) + \frac{\partial}{\partial q_2}(\frac{h_1 h_3}{h_2}\frac{\partial}{\partial q_2}) + \frac{\partial}{\partial q_3}(\frac{h_1 h_2}{h_3}\frac{\partial}{\partial q_3})\right]}.$$

Cartesian: $(q_1, q_2, q_3) = (x, y, z)$ and $(h_1, h_2, h_3) = (1,1,1)$.

$$\nabla^2 = \frac{1}{1\cdot 1\cdot 1}\left[\frac{\partial}{\partial x}(\frac{1\cdot 1}{1}\frac{\partial}{\partial y}) + \frac{\partial}{\partial q_2}(\frac{1\cdot 1}{1}\frac{\partial}{\partial q_2}) + \frac{\partial}{\partial q_3}(\frac{1\cdot 1}{1}\frac{\partial}{\partial q_3})\right]$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Cylindrical: $(q_1, q_2, q_3) = (\rho, \phi, z)$ and $(h_1, h_2, h_3) = (1, \rho, 1)$.

$$\nabla^2 = \frac{1}{1\cdot \rho \cdot 1}\left[\frac{\partial}{\partial \rho}(\frac{\rho\cdot 1}{1}\frac{\partial}{\partial \rho}) + \frac{\partial}{\partial \phi}(\frac{1\cdot 1}{\rho}\frac{\partial}{\partial \phi}) + \frac{\partial}{\partial z}(\frac{1\cdot \rho}{1}\frac{\partial}{\partial z})\right]$$

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho\frac{\partial}{\partial \rho}) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

Spherical: $(q_1, q_2, q_3) = (r, \theta, \phi)$ and $(h_1, h_2, h_3) = (1, r, r\sin\theta)$.

$$\nabla^2 = \frac{1}{1\cdot r\cdot r\sin\theta}\left[\frac{\partial}{\partial r}(\frac{r\cdot r\sin\theta}{1}\frac{\partial}{\partial r}) + \frac{\partial}{\partial \theta}(\frac{1\cdot r\sin\theta}{r}\frac{\partial}{\partial \theta}) + \frac{\partial}{\partial \phi}(\frac{1\cdot r}{r\sin\theta}\frac{\partial}{\partial \phi})\right]$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial}{\partial \theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

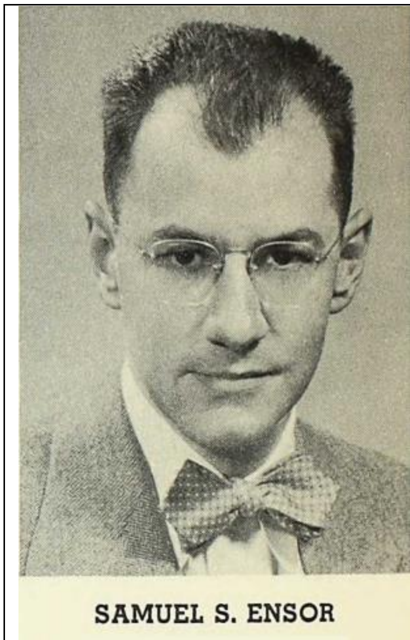


Photo Courtesy Ancestry.com, scan from the *Greatonion*, SJ 1953 Yearbook

Many years ago Mr. Samuel S. Ensor, my Calculus teacher at St. Joseph's College (SJ) in Philadelphia (University since 1978) gave us a project in Calculus III that was long, but very useful and productive (Spring 1969). It is given below. Everyone aspiring to be a physicist or engineer should do this calculation once sometime in their studies. It will correct any rough edges you have in taking partial derivatives and using the chain rule.

Recommended Problem for a Semester Break. Derive the Laplacian in spherical coordinates the long way! Start with

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and

$$x = r\sin\theta\cos\phi\,,\; y = r\sin\theta\sin\phi\,,\; z = r\cos\theta\,.$$

So you begin cranking away with

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}$$ and so on.

Have fun!

Q8. Bonus: Two Maxwell Equations in Differential Form.

$$\oiint \vec{E}\cdot\vec{dA} = \iiint\left[\nabla\cdot\vec{E}\right]dV \qquad \oiint \vec{B}\cdot\vec{dA} = \iiint\left[\nabla\cdot\vec{B}\right]dV$$

$$\oiint \vec{B}\cdot\vec{dA} = 0 \quad \Rightarrow \quad \nabla\cdot\vec{B} = 0$$

$$\oiint \vec{E}\cdot\vec{dA} = \frac{Q}{\varepsilon_o} = \frac{1}{\varepsilon_o}\iiint \rho\, dV \quad \Rightarrow \quad \nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$