R1. Fourier Series.

Jean Baptiste Joseph Fourier (1768-1830)
Courtesy School of Mathematics and Statistics
University of St. Andrews, Scotland

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left[a_m \cos(mx) + b_m \sin(mx)\right]$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin(nx)\,dx$$

Derivations follow.

(1) First consider $\int_{-\pi}^{+\pi} f(x)\,dx$.

$$\int_{-\pi}^{+\pi} f(x)\,dx = \int_{-\pi}^{+\pi}\frac{a_0}{2}dx + \sum_{m=1}^{\infty}\left[\int_{-\pi}^{+\pi} a_m \cos(mx)\,dx + \int_{-\pi}^{+\pi} b_m \sin(mx)\,dx\right]$$

Question: Why can we write $\int_{-\pi}^{+\pi}\cos(mx)\,dx = 0$ and $\int_{-\pi}^{+\pi}\sin(mx)\,dx = 0$?

Answer: Think area under these functions over a single cycle.

Therefore, the trig functions don't appear:

$$\int_{-\pi}^{+\pi} f(x)\,dx = \int_{-\pi}^{+\pi}\frac{a_0}{2}dx = \frac{a_0}{2}\,x\Big|_{-\pi}^{+\pi} = \frac{a_0}{2}\left[\pi-(-\pi)\right] = a_0\pi .$$

$$\boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\,dx}$$

(2) Second consider

$$\int_{-\pi}^{+\pi} f(x)\cos(nx)dx = \int_{-\pi}^{+\pi} \frac{a_0}{2}\cos(nx)dx$$

$$+\sum_{m=1}^{\infty}\left[\int_{-\pi}^{+\pi} a_m \cos(nx)\cos(mx)dx + \int_{-\pi}^{+\pi} b_m \cos(nx)\sin(mx)dx\right]$$

The first integral is zero: $\int_{-\pi}^{+\pi} \frac{a_0}{2}\cos(nx)dx = 0$.

Case $n \neq m$ for the 2nd integral.

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$\int_{-\pi}^{+\pi}\cos(nx)\cos(mx)dx = \int_{-\pi}^{+\pi}\left(\frac{e^{inx}+e^{-inx}}{2}\right)\left(\frac{e^{imx}+e^{-imx}}{2}\right)dx$$

Four pieces are present, each having the form below with integer $p \neq 0$.

$$\int_{-\pi}^{+\pi} e^{ipx}dx = \left.\frac{e^{ipx}}{ip}\right|_{-\pi}^{+\pi} = \frac{1}{ip}\left[\cos(px)+i\sin(px)\right]\Big|_{-\pi}^{+\pi}$$

$$\int_{-\pi}^{+\pi} e^{ipx}dx = \left.\frac{e^{ipx}}{ip}\right|_{-\pi}^{+\pi} = \frac{1}{ip}\left[\cos(p\pi)+i\sin(p\pi)\right] - \frac{1}{ip}\left[\cos(-p\pi)+i\sin(-p\pi)\right]$$

$$\int_{-\pi}^{+\pi} e^{ipx}dx = \frac{1}{ip}\left[-1+i\cdot 0\right] - \frac{1}{ip}\left[-1+i\cdot 0\right] = 0$$

Case $n = m$ for the 2nd integral. Over a full cycle or complete number of cycles

$$\int_{-\pi}^{+\pi}\cos^2(nx)\,dx = \int_{-\pi}^{+\pi}\sin^2(nx)\,dx = \frac{1}{2}\left[\int_{-\pi}^{+\pi}\cos^2(nx)\,dx + \int_{-\pi}^{+\pi}\sin^2(nx)\,dx\right]$$

$$\int_{-\pi}^{+\pi} \cos^2(nx)\,dx = \frac{1}{2}\left[\int_{-\pi}^{+\pi} \cos^2(nx) + \sin^2(nx)\right] dx$$

$$\int_{-\pi}^{+\pi} \cos^2(nx)\,dx = \frac{1}{2}\int_{-\pi}^{+\pi} dx = \int_0^{\pi} dx = x\Big|_0^{\pi} = \pi - 0 = \pi$$

Summary: $\int_{-\pi}^{+\pi} \cos(nx)\cos(mx)dx = \pi\delta_{nm}$, where

$$\delta_{ij} = 0 \text{ if } i \neq j$$

$$\delta_{ij} = 1 \text{ if } i = j.$$

The symbol δ_{ij} is called the Kronecker delta.

What about $\int_{-\pi}^{+\pi} \cos(nx)\sin(mx)dx$?

Since we have a product of an even function and an odd one over a symmetric interval

$$\int_{-\pi}^{+\pi} \cos(nx)\sin(mx)dx = 0$$

Getting back to our consideration of $\int_{-\pi}^{+\pi} f(x)\cos(nx)dx$, we have the result

$$\int_{-\pi}^{+\pi} f(x)\cos(nx)dx = \int_{-\pi}^{+\pi} \frac{a_0}{2}\cos(nx)dx$$

$$+\sum_{m=1}^{\infty}\left[\int_{-\pi}^{+\pi} a_m \cos(nx)\cos(mx)dx + \int_{-\pi}^{+\pi} b_m \cos(nx)\sin(mx)dx\right]$$

$$\int_{-\pi}^{+\pi} f(x)\cos(nx)dx = 0 + \sum_{m=1}^{\infty}(a_m\pi\delta_{nm} + b_m \cdot 0) = a_n\pi$$

$$\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos(nx)dx}$$

(3) Third consider

$$\int_{-\pi}^{+\pi} f(x)\sin(nx)dx = \int_{-\pi}^{+\pi} \frac{a_0}{2}\cos(nx)dx$$

$$+\sum_{m=1}^{\infty}\left[\int_{-\pi}^{+\pi} a_m \sin(nx)\cos(mx)dx + \int_{-\pi}^{+\pi} b_m \sin(nx)\sin(mx)dx\right]$$

A similar analysis as we did for the cosine leads to

$$\int_{-\pi}^{+\pi} \sin(nx)\sin(mx)dx = \pi\delta_{nm}\,.$$

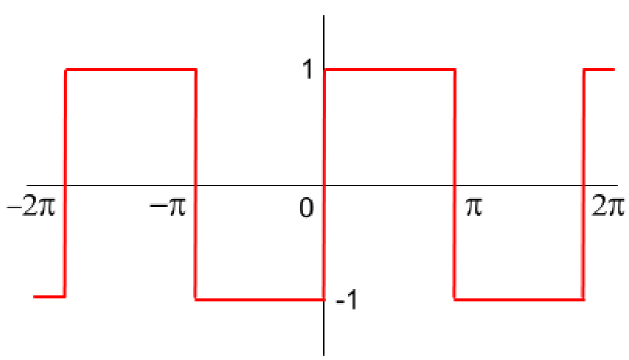
Then $\int_{-\pi}^{+\pi} f(x)\sin(nx)dx = 0 + \sum_{m=1}^{\infty}(a_m \cdot 0 + b_m \cdot \pi\delta_{nm}) = b_n\pi$

$$\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin(nx)dx}$$

All has been derived!

R2. Series Example: The Square Wave.

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left[a_m \cos(mx) + b_m \sin(mx)\right]$$



Since the above square wave is an odd function, the a_0 and a_n integrals are zero. The b_n integral is the one that will give nonzero values.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin(nx)dx = \frac{2}{\pi}\int_0^{\pi}\sin(nx)dx$$

$$b_n = -\frac{2}{\pi}\frac{\cos(nx)}{n}\bigg|_0^{\pi} = -\frac{2}{\pi}\frac{1}{n}\left[\cos(n\pi) - \cos(0)\right]$$

For even $n = 2k$, where $k = 1, 2, 3...$

$$b_{2k} = -\frac{2}{\pi}\frac{1}{2k}\left[\cos(2\pi k) - \cos(0)\right] = -\frac{2}{\pi}\frac{1}{2k}(1-1) = 0$$

For odd $n = 2k-1$, where $k = 1, 2, 3...$

$$b_{2k-1} = -\frac{2}{\pi}\frac{1}{2k-1}\left[\cos(2\pi k - \pi) - \cos(0)\right]$$

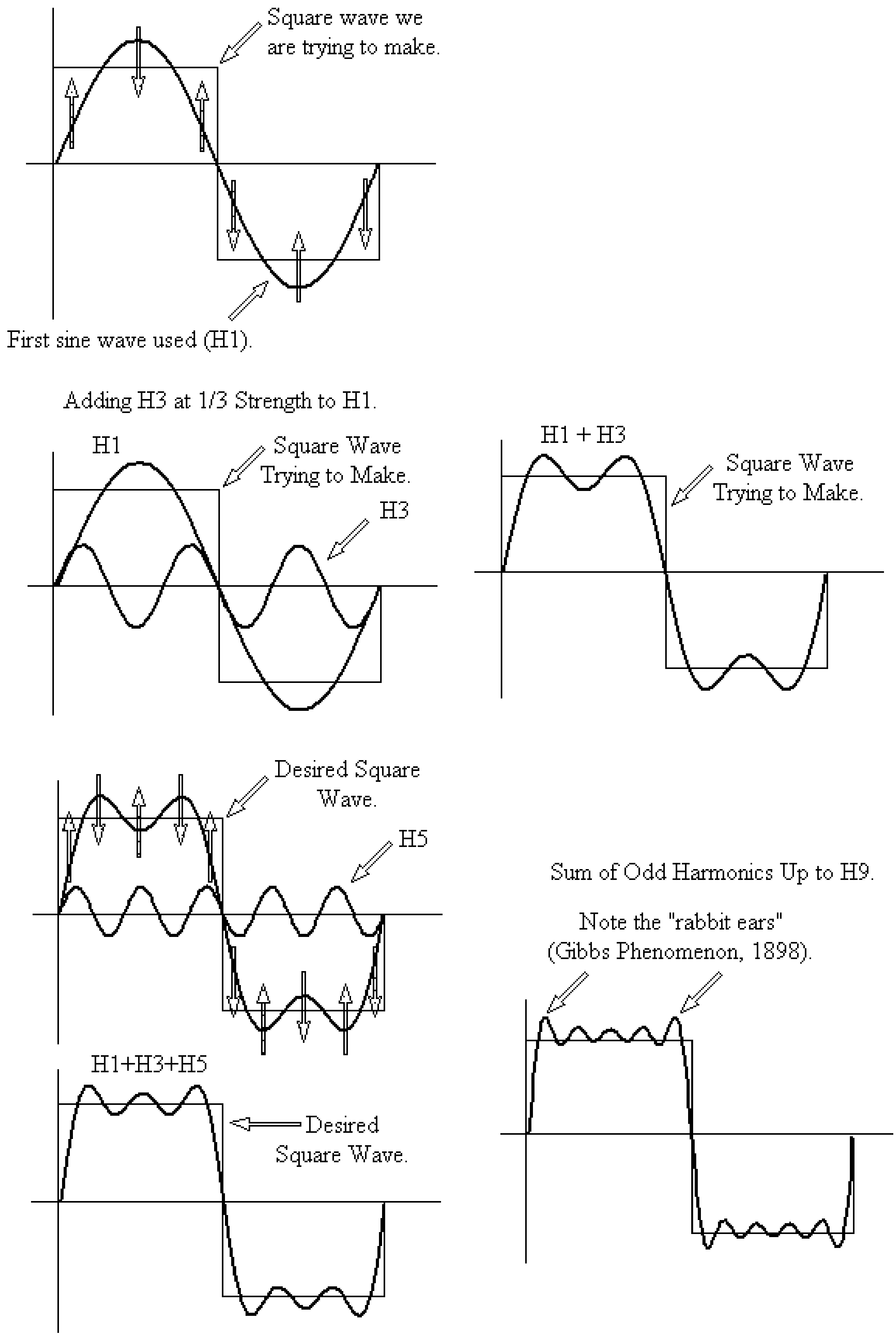
$$= -\frac{2}{\pi}\frac{1}{2k-1}(-1-1) = \frac{4}{\pi}\frac{1}{2k-1}$$

$$b_n = \frac{4}{\pi}\frac{1}{n} \text{ for odd } n.$$

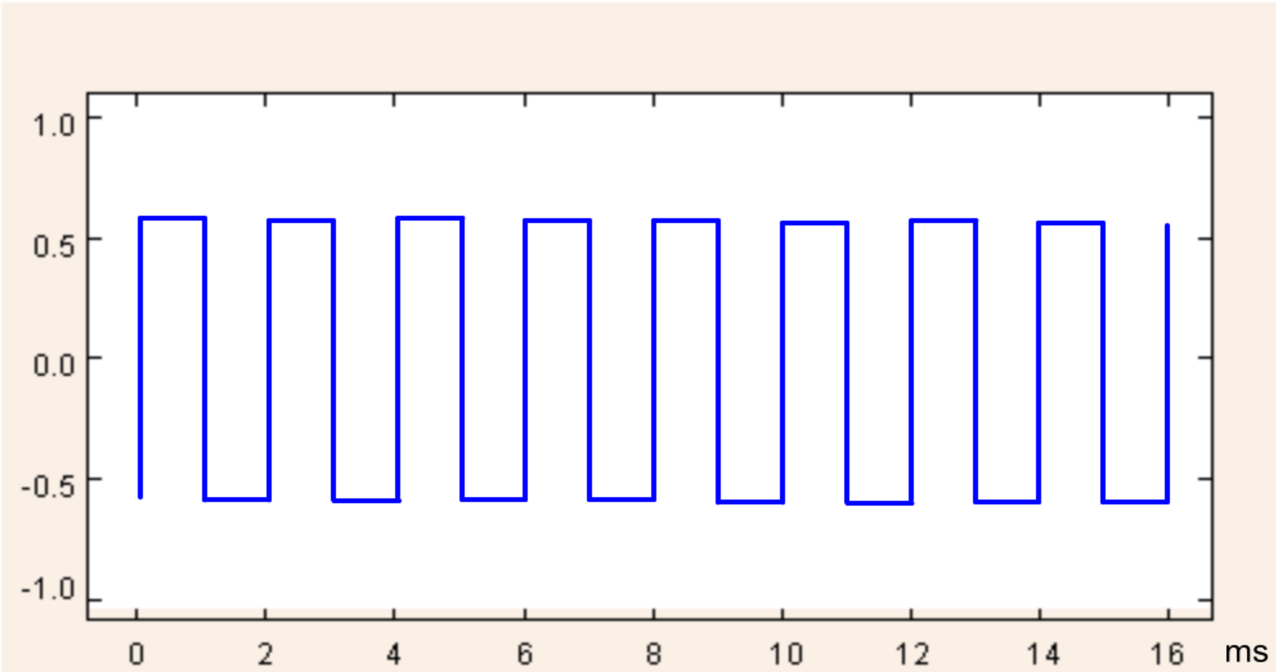
$$f(x) = \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x)...\right]$$

Michael J. Ruiz, "Free Sixteen Harmonic Fourier Series Web App with Sound," *Physics Education* **53**, 025008 (March 2018). **pdf** and **The App**

Visualizations Follow

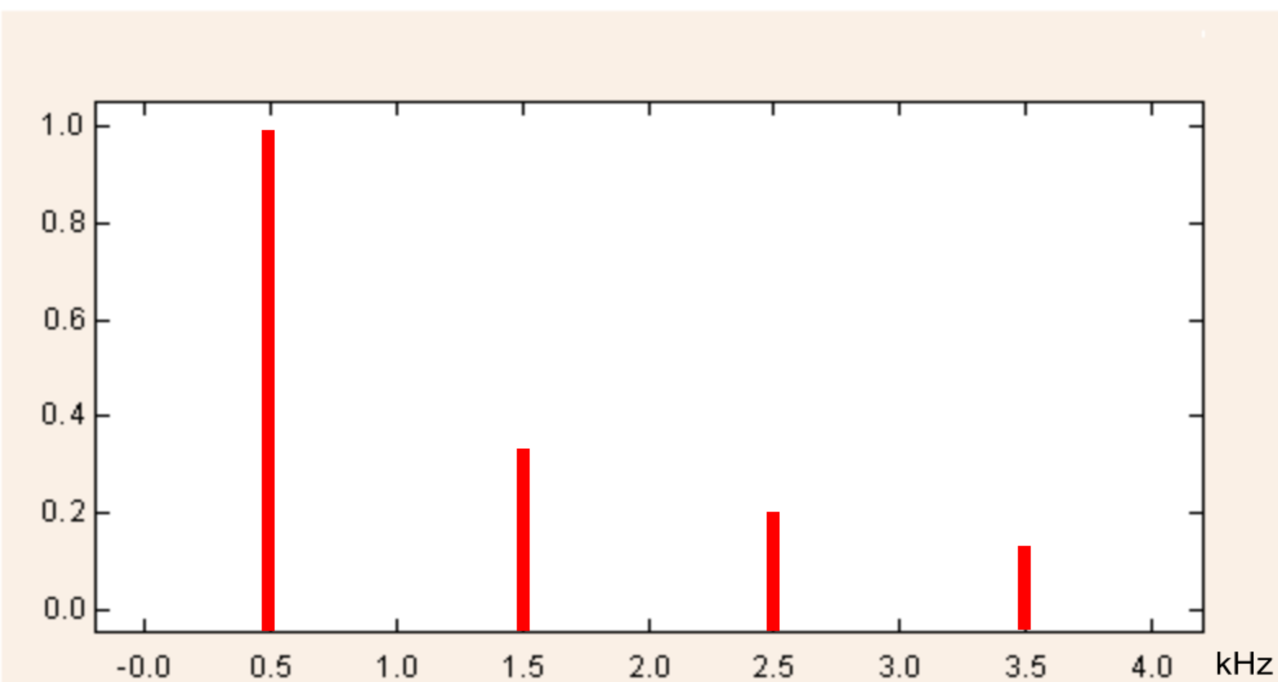

Square wave we are trying to make.
First sine wave used (H1).
Adding H3 at 1/3 Strength to H1.
H1
Square Wave Trying to Make.
H3
H1 + H3
Square Wave Trying to Make.
Desired Square Wave.
H5
H1+H3+H5
Desired Square Wave.
Sum of Odd Harmonics Up to H9.
Note the "rabbit ears" (Gibbs Phenomenon, 1898).

Time Domain



Adapted from the University of California, Berkeley

Frequency Domain



Adapted from the University of California, Berkeley

R3. Fourier Transform.

Remember our wave packet $\psi(x,t) = \int A(k) e^{i(kx - \omega t)} dk$?

Watch these math steps that will lead to an integral like $\int A(k) e^{ikx} dk$.

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left[a_m \cos(mx) + b_m \sin(mx) \right]$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin(nx) dx$$

$$\cos(mx) = \frac{e^{imx} + e^{-imx}}{2} \quad \text{and} \quad \sin(mx) = \frac{e^{imx} - e^{-imx}}{2i} = \frac{ie^{-imx} - ie^{imx}}{2}$$

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left[a_m \left(\frac{e^{imx} + e^{-imx}}{2} \right) + b_m \left(\frac{ie^{-imx} - ie^{imx}}{2} \right) \right]$$

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left[\left(\frac{a_m - ib_m}{2} \right) e^{imx} + \left(\frac{a_m + ib_m}{2} \right) e^{-imx} \right]$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_0 = \frac{a_0}{2} \qquad c_n = \frac{1}{2}(a_n - ib_n) \;\; \text{for} \;\; n > 0 \qquad c_n = \frac{1}{2}(a_n + ib_n) \;\; \text{for} \;\; n < 0$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x)\, dx \quad \Rightarrow \quad c_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x)\, dx$$

Now consider the above c_n definitions with

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos(nx)dx \quad \text{and} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin(nx)dx .$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\left[\cos(nx) - i\sin(nx)\right]dx \quad \text{for} \quad n > 0 .$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)e^{-inx}dx$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\left[\cos(nx) + i\sin(nx)\right]dx \quad \text{for} \quad n < 0 .$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)e^{+inx}dx .$$

Therefore, for all n, positive, negative, or even zero, the following works.

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)e^{-inx}dx$$

For notational purposes, rewrite the above with a new z-variable and g(z).

$$\boxed{c_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(z)e^{-inz}dz} \qquad \boxed{f(z) = \sum_{n=-\infty}^{\infty} c_n e^{inz}}$$

Then expand the interval by a transformation of variables.

$$-\pi \leq z \leq +\pi$$

$$-L \leq x \leq +L$$

This leads us to $\frac{z}{x} = \frac{\pi}{L}$, i.e., $z = \frac{\pi}{L}x$ and $dz = \frac{\pi}{L}dx$. We arrive at

$$g(\frac{\pi}{L}x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{L}x} \quad \text{and} \quad c_n = \frac{1}{2\pi}\int_{-L}^{+L} g(\frac{\pi}{L}x) e^{-i\frac{n\pi}{L}x} \frac{\pi}{L}dx .$$

We can write these as follows.

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{L}x}} \qquad \boxed{c_n = \frac{1}{2L}\int_{-L}^{+L} f(x) e^{-i\frac{n\pi}{L}x} dx}$$

It is time for some "theoretical physics" magic. Note that this chapter is not meant to be super mathematically rigorous, as physics usually never is. Our focus here is trying to understand where the Fourier transform comes from rather than just giving it to you.

We would like to transform the series to an integral. We note that since the n are integers, then $\Delta n = 1$. We then write

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{L}x} \Delta n .$$

Now we introduce a new variable, one which we intend to promote to a continuous variable.

$$k = \frac{n\pi}{L} .$$

Therefore,

$$\Delta k = \frac{\pi}{L}\Delta n \quad \text{and} \quad \Delta n = \frac{L}{\pi}\Delta k \text{, leading to}$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ikx} \frac{L}{\pi}\Delta k .$$

Let $Lc_n \to c(k)$. With the new variable k now a continuous variable,

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} c(k)e^{ikx}dk \, .$$

The variable k has become a continuous variable and we have replaced the sum with an integral. The three things we did: 1) replace delta k with dk, 2) "rip off" the n from the c and introduce c(k), and 3) turn the summation sign into a "snake" where we integrate over all k since our sum did that for the discrete case.

What about our other equation?

$$c_n = \frac{1}{2L}\int_{-L}^{+L} f(x)e^{-i\frac{n\pi}{L}x}dx$$

With our new variable we have

$$Lc_n \to c(k) = \frac{1}{2}\int_{-L}^{+L} f(x)e^{-ikx}dx \, .$$

Summary:

$$\boxed{f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} c(k)e^{ikx}dk}$$

$$\boxed{c(k) = \frac{1}{2}\int_{-\infty}^{+\infty} f(x)e^{-ikx}dx}$$

We will adopt the following convention for defining the Fourier Transform. Our convention will involve a symmetric definition, but you do not have to things this way. The convention we will use is to define

$$c(k) = \sqrt{\frac{\pi}{2}}F(k) \, .$$

Then, our two equations

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} c(k)e^{ikx}dk \quad \text{and} \quad c(k) = \frac{1}{2}\int_{-\infty}^{+\infty} f(x)e^{-ikx}dx$$

become

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} c(k)e^{ikx}dk = \frac{1}{\pi}\int_{-\infty}^{\infty} \sqrt{\frac{\pi}{2}}F(k)e^{ikx}dk$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{ikx}dk$$

$$\sqrt{\frac{\pi}{2}}F(k) = \frac{1}{2}\int_{-\infty}^{+\infty} f(x)e^{-ikx}dx \quad => \quad F(k) = \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{+\infty} f(x)e^{-ikx}dx$$

The result is the symmetric equations below.

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{ikx}dk} \qquad \boxed{F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-ikx}dx}$$

The function $F(k)$ is called the Fourier transform of $f(x)$ and $f(x)$ is the inverse Fourier transform of $F(k)$.

Some authors write the Fourier transform with the following notation.

$$\Im\{f(x)\} \equiv F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-ikx}dx$$

The inverse Fourier transform is then

$$\Im^{-1}\{F(k)\} \equiv f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{ikx}dk$$

There's your $\int A(k)e^{ikx}dk$ as promised earlier!

R4. Transform Example: Rectangular Pulse.

$f(x) = 1$ for $-\frac{a}{2} \le x \le \frac{a}{2}$ where $a > 0$ and $f(x) = 0$ elsewhere.

$$\Im\{f(x)\} \equiv F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-a/2}^{+a/2} e^{-ikx} dx$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \frac{e^{-ikx}}{(-ik)} \Bigg|_{-a/2}^{+a/2}$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \frac{\left[e^{-ika/2} - e^{ika/2} \right]}{(-ik)}$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \frac{\left[e^{ika/2} - e^{-ika/2} \right]}{ik}$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \frac{2}{k} \left[\frac{e^{ika/2} - e^{-ika/2}}{2i} \right]$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \frac{2a}{ka} \sin(\frac{ka}{2})$$

$$F(k) = \frac{2a}{\sqrt{2\pi}} \frac{\sin(ka/2)}{ka/2} = \frac{2a}{\sqrt{2\pi}} \text{sinc}(\frac{ka}{2})$$

The sinc function is defined as $\operatorname{sinc}(x) = \dfrac{\sin x}{x}$.

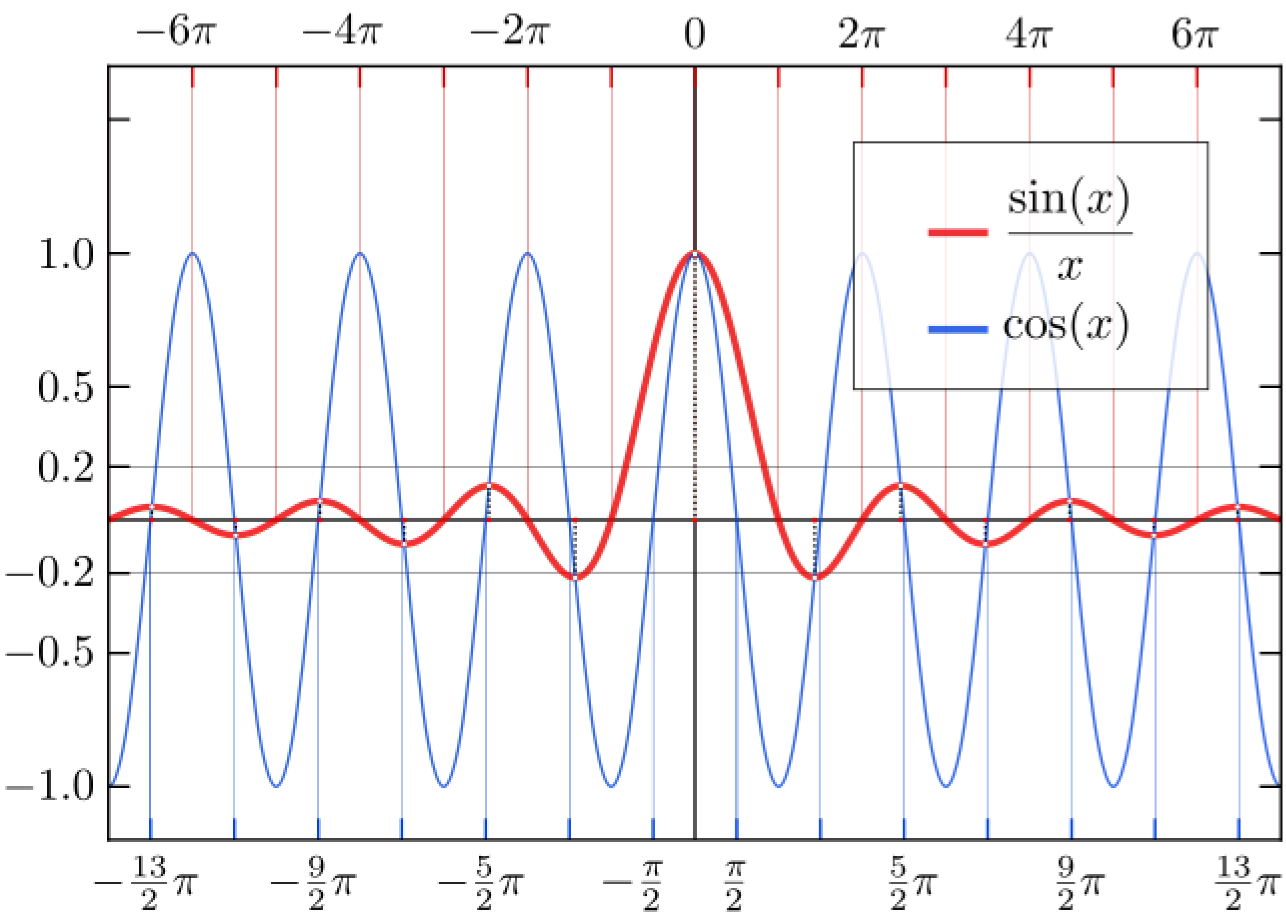
Now it's time to remember L'Hôpital's rule (also L'Hospital's rule).

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

Applying L'Hôpital's rule,

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \cos 0 = 1.$$

The sinc(x) function is compared below to cos(x).



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