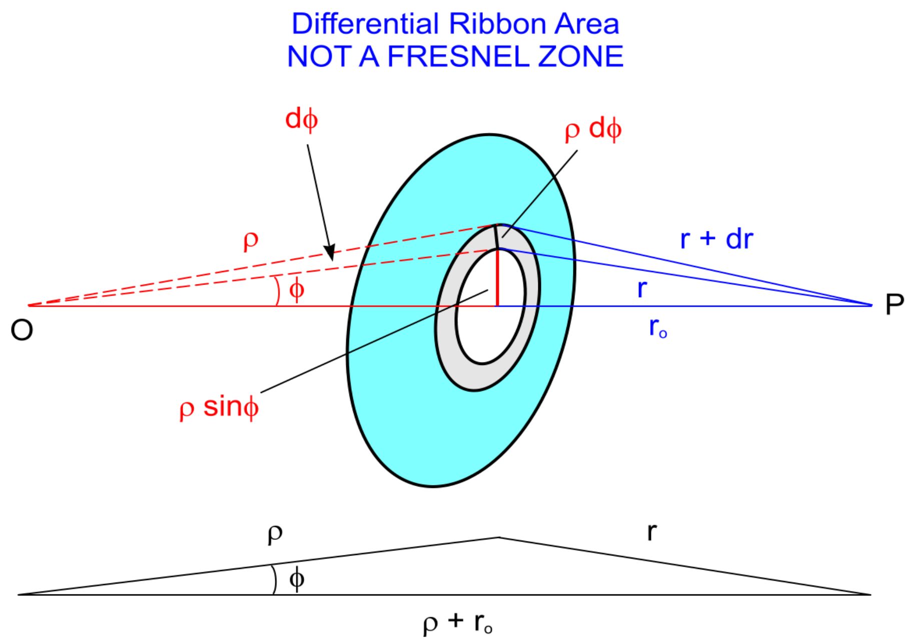
HW U1. Differential Ribbon Area.



Use calculus to set up the area of the gray ribbon with thickness $\rho d\phi$ and radius $\rho \sin\phi$ and show that the differential area of the ribbon is

$$dA = \frac{2\pi\rho}{(\rho + r_o)} r dr$$

following the steps below.

(a) First express dS in terms of ρ, ϕ, and $d\phi$ as done in calculus and intro physics, e.g., when you calculated areas of disks or integrated moments of inertia.

(b) Use the law of cosines for the triangle in the figure to relate ρ, r_o, r, and $\cos\phi$. Get r^2 on one side of the equation.

(c) Use calculus on your result in (b) to show that $2rdr = 2\rho(\rho + r_o)\sin\phi\, d\phi$

(d) In your expression found in (a) replace the $d\phi$ using your formula in (c).

For full credit you must show all steps.

SOLUTION (a) From math or physics class: Ribbon area is $\boxed{dA = (2\pi\rho\sin\phi)(\rho d\phi)}$.

(b) Law of cosines: $\boxed{r^2 = \rho^2 + (\rho + r_o)^2 - 2\rho(\rho + r_o)\cos\phi}$

(c) $2rdr = 0 + 0 - 2\rho(\rho + r_o)[-\sin\phi]d\phi$

$$\boxed{2rdr = 2\rho(\rho + r_o)\sin\phi d\phi}$$

(d) $2\rho(\rho + r_o)\sin\phi d\phi = 2rdr$ => $\sin\phi d\phi = \frac{1}{\rho(\rho + r_o)} rdr$

$$dA = (2\pi\rho\sin\phi)(\rho d\phi) = 2\pi\rho^2 \sin\phi d\phi$$

$$dA = 2\pi\rho^2\left[\sin\phi\, d\phi\right] = 2\pi\rho^2\left[\frac{1}{\rho(\rho + r_o)} r\,dr\right]$$

$$\boxed{dA = \frac{2\pi\rho}{(\rho + r_o)} r\,dr}$$

HU2. Fresnel Zone Part II. Use your $dA = \frac{2\pi\rho}{(\rho + r_o)} r\,dr$ from HW U1 to find the area of the m[th] Fresnel zone by integrating r from $r_o + (m-1)\frac{\lambda}{2}$ to $r_o + m\frac{\lambda}{2}$. From this integration, show that the area of the m[th] Fresnel zone is given by

$$A_m = \frac{\lambda\pi\rho}{\rho + r_o}\left[r_o + \frac{(2m-1)\lambda}{4}\right].$$

For full credit you must show all steps of the integration.

SOLUTION

$$dA = \frac{2\pi\rho}{(\rho + r_o)} r\,dr$$

$$A_m = \int_{r_o + (m-1)\frac{\lambda}{2}}^{r_o + m\frac{\lambda}{2}} \frac{2\pi\rho}{(\rho + r_o)} r\,dr \quad \Rightarrow \quad A_m = \frac{2\pi\rho}{(\rho + r_o)} \int_{r_o + (m-1)\frac{\lambda}{2}}^{r_o + m\frac{\lambda}{2}} r\,dr$$

$$A_m = \frac{2\pi\rho}{(\rho + r_o)} \frac{r^2}{2}\Bigg|_{r_o + (m-1)\frac{\lambda}{2}}^{r_o + m\frac{\lambda}{2}} \quad \Rightarrow \quad \frac{(\rho + r_o)}{\pi\rho} A_m = r^2\Bigg|_{r_o + (m-1)\frac{\lambda}{2}}^{r_o + m\frac{\lambda}{2}}$$

$$\frac{(\rho + r_o)}{\pi\rho} A_m = \left[r_o + m\frac{\lambda}{2}\right]^2 - \left[r_o + (m-1)\frac{\lambda}{2}\right]^2$$

$$\frac{(\rho + r_o)}{\pi\rho} A_m = \left[r_o^2 + r_o m\lambda + m^2 \frac{\lambda^2}{4} \right] - \left[r_o^2 + r_o (m-1)\lambda + \frac{(m-1)^2 \lambda^2}{4} \right]$$

$$\frac{(\rho + r_o)}{\pi\rho} A_m = r_o^2 + r_o m\lambda + m^2 \frac{\lambda^2}{4} - r_o^2 - r_o (m-1)\lambda - \frac{(m-1)^2 \lambda^2}{4}$$

$$\frac{(\rho + r_o)}{\pi\rho} A_m = r_o m\lambda + m^2 \frac{\lambda^2}{4} - r_o (m-1)\lambda - \frac{(m-1)^2 \lambda^2}{4}$$

$$\frac{(\rho + r_o)}{\pi\rho} A_m = r_o m\lambda + m^2 \frac{\lambda^2}{4} - r_o m\lambda + r_o\lambda - (m^2 - 2m + 1) \frac{\lambda^2}{4}$$

$$\frac{(\rho + r_o)}{\pi\rho} A_m = m^2 \frac{\lambda^2}{4} + r_o\lambda - (m^2 - 2m + 1) \frac{\lambda^2}{4}$$

$$\frac{(\rho + r_o)}{\pi\rho} A_m = r_o\lambda + (m^2 - m^2 + 2m - 1) \frac{\lambda^2}{4}$$

$$\frac{(\rho + r_o)}{\pi\rho} A_m = r_o\lambda + (2m - 1) \frac{\lambda^2}{4}$$

$$\frac{(\rho + r_o)}{\pi\rho} A_m = \lambda \left[r_o + (2m - 1) \frac{\lambda}{4} \right]$$

$$\boxed{A_m = \frac{\lambda\pi\rho}{\rho + r_o} \left[r_o + \frac{(2m-1)\lambda}{4} \right]}$$