J0. Inelastic and Elastic Collisions. We revisit the collision in the previous chapter with the skaters. We employed conservation of momentum $m_1 u_1 = (m_1 + m_2)v$, where the lady skated at speed u_1, collided with her partner at rest u_2, then they continued together at speed v.

Photo Courtesy Stephen Downes, flickr. [License: Attribution-NonCommercial 2.0]

We will now show that kinetic energy is not conserved in this case. The kinetic energy before is

$$K_{before} = \frac{1}{2} m_1 u_1^2 .$$

The kinetic energy afterwards is

$$K_{after} = \frac{1}{2}(m_1 + m_2)v^2 .$$

We can relate v to u_1 via the momentum equation $m_1 u_1 = (m_1 + m_2)v$.

$$v = \frac{m_1}{m_1 + m_2} u_1$$

Substituting in $K_{after} = \frac{1}{2}(m_1 + m_2)v^2$ gives us

$$K_{after} = \frac{1}{2}(m_1 + m_2)(\frac{m_1}{m_1 + m_2} u_1)^2$$

$$K_{after} = \frac{1}{2}(m_1 + m_2)\frac{m_1^2 u_1^2}{(m_1 + m_2)^2}$$

$$K_{after} = \frac{1}{2}\frac{m_1^2 u_1^2}{m_1 + m_2} ,$$

which is less than $K_{before} = \frac{1}{2} m_1 u_1^2$.

The ratio K_{after} / K_{before} is

$$\frac{K_{after}}{K_{before}} = \frac{\frac{1}{2}\frac{m_1^2 u_1^2}{m_1 + m_2}}{\frac{1}{2} m_1 u_1^2}$$

$$\frac{K_{after}}{K_{before}} = \frac{m_1}{m_1 + m_2}$$

Kinetic energy is not conserved. We say that the collision is *inelastic*. For the masses of our skaters given in the last chapter: $m_1 = 50.0 \text{ kg}$ and $m_2 = 75.0 \text{ kg}$, we find

$$\frac{K_{after}}{K_{before}} = \frac{m_1}{m_1 + m_2} = \frac{50}{50 + 75} = \frac{50/25}{50/25 + 75/25} = \frac{2}{2+3} = \frac{2}{5} .$$

The kinetic energy afterwards is 40% of what it was before the skaters interacted. If the first mass is super small, the second mass will stop it during the collision.

$$\lim_{m_1 \to 0} \frac{K_{after}}{K_{before}} = \lim_{m_1 \to 0} \frac{m_1}{m_1 + m_2} = \frac{0}{0 + m_2} = 0 \qquad \lim_{m_1 \to 0} v = \lim_{m_1 \to 0} \frac{m_1}{m_1 + m_2} u_1 = \frac{0}{0 + m_2} u_1 = 0$$

If the second mass is very small, the first mass will grab it and proceed unimpeded. Technically, the combination will travel slightly less than the speed of the incoming mass, i.e., the first mass.

$$\lim_{m_2 \to 0} \frac{K_{after}}{K_{before}} = \lim_{m_2 \to 0} \frac{m_1}{m_1 + m_2} = \frac{m_1}{m_1 + 0} = 1 \qquad \lim_{m_2 \to 0} v = \lim_{m_2 \to 0} \frac{m_1}{m_1 + m_2} u_1 = \frac{m_1}{m_1 + 0} u_1 = u_1$$

Did we increase the kinetic energy since now both masses move at the initial speed? No, because in the limit the second mass $m_2 \to 0$, i.e., vanishes, and contributes nothing.

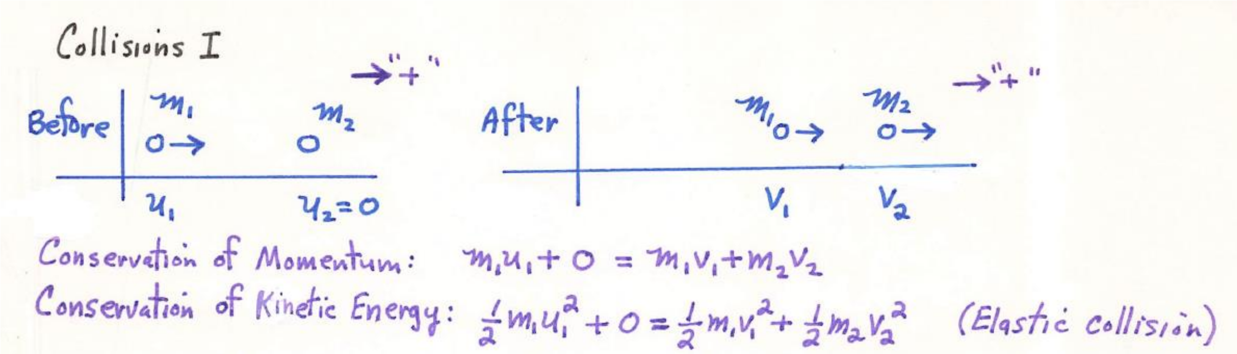
Where did the energy for the skaters go? The kinetic energy lost went into the bodies of the skaters. They absorbed the extra energy.

The two types of collisions are

elastic collision – kinetic energy is conserved,
inelastic collision – kinetic energy is not conserved.

In both cases energy is conserved, meaning the total energy. When the two masses get connected, one also calls that case a *perfectly inelastic collision*.

J1. Elastic Collisions I. We consider a 1-dimensional collision where mass m_1 collides with m_2.



There are no external forces. Think of doing the experiment in outer space. But it will turn out that our formulas apply quite well to examples on Earth, such as billiards. There, the external force of gravity pulls downward and does no work on the horizontally moving balls.

No external forces means we can apply conservation of momentum. We will also assume elastic collisions, which means all the energy stays in the form of kinetic energy when the masses separate after contact. Such collisions allow us to apply conservation of kinetic energy.

In this section only the first mass moves initially. The second mass is at rest. In the next section, we will analyze the situation where both masses have initial velocities.

The equations are:

conservation of momentum: $m_1u_1 + 0 = m_1v_1 + m_2v_2$,

conservation of kinetic energy: $\frac{1}{2}m_1u_1^2 + 0 = \frac{1}{2}m_1v_1^2 + m_2v_2^2$,

where "u" stands for an initial velocity and "v" for a final. I use this convention since I hate subscripts and do not want to include "i" for initial and "f" for final in addition to "1" and "2" as subscripts. I pick u for initial (before) and v for final (after) since u comes before v in the alphabet.

What is of interest? That would be the final velocities. I am going to work with pairs of equation and will bracket them like a friend of mine did in grad school.

$$\left\{\begin{matrix} m_1u_1 + 0 = m_1v_1 + m_2v_2 \\ \frac{1}{2}m_1u_1^2 + 0 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \end{matrix}\right\}$$

Multiply the second equation by 2.

$$\begin{Bmatrix} m_1u_1 = m_1v_1 + m_2v_2 \\ m_1u_1^2 = m_1v_1^2 + m_2v_2^2 \end{Bmatrix}$$

Now bring terms with the subscript 1 to the left side in each case.

$$\begin{Bmatrix} m_1(u_1 - v_1) = m_2v_2 \\ m_1(u_1^2 - v_1^2) = m_2v_2^2 \end{Bmatrix}$$

Next we factor the difference of squares using the identity $a^2 - b^2 = (a+b)(a-b)$.

$$\begin{Bmatrix} m_1(u_1 - v_1) = m_2v_2 \\ m_1(u_1 + v_1)(u_1 - v_1) = m_2v_2^2 \end{Bmatrix}$$

We make a big discovery by staring at these equations. Why must the following be true?

$$(u_1 + v_1) = v_2$$

If you do not see it, what the doctorphys video on YouTube for this section.

With the initial speed on the left side, we find the next equation.

$$\boxed{u_1 = v_2 - v_1}$$

In words, this equation says *the velocity of approach equals the velocity of separation*.

Using $(u_1 + v_1) = v_2$ as our second equation, our two equations are now

$$\begin{Bmatrix} m_1(u_1 - v_1) = m_2v_2 \\ u_1 + v_1 = v_2 \end{Bmatrix}.$$

Eliminating v_2 in the first equation, using the second,

$$m_1(u_1 - v_1) = m_2(u_1 + v_1)$$

We want v_1 on one side.

$$m_1u_1 - m_1v_1 = m_2u_1 + m_2v_1$$

$$m_1u_1 - m_2u_1 = m_1v_1 + m_2v_1$$

$$(m_1 - m_2)u_1 = (m_1 + m_2)v_1$$

$$\boxed{v_1 = (\frac{m_1 - m_2}{m_1 + m_2})u_1}$$

What remains now is to find the formula for v_2.

We can use $(u_1 + v_1) = v_2$ from above.

Substitute $v_1 = (\frac{m_1 - m_2}{m_1 + m_2})u_1$ in $v_2 = u_1 + v_1$.

$$v_2 = u_1 + (\frac{m_1 - m_2}{m_1 + m_2})u_1$$

$$v_2 = \left[1 + (\frac{m_1 - m_2}{m_1 + m_2})\right]u_1 \quad => \quad v_2 = \left[\frac{m_1 + m_2}{m_1 + m_2} + \frac{m_1 - m_2}{m_1 + m_2}\right]u_1$$

$$v_2 = \left[\frac{m_1 + m_2 + m_1 - m_2}{m_1 + m_2}\right]u_1 \quad => \quad v_2 = \left[\frac{2m_1}{m_1 + m_2}\right]u_1$$

Our two final velocity equations are below.

$$\boxed{v_1 = (\frac{m_1 - m_2}{m_1 + m_2})u_1} \qquad \boxed{v_2 = (\frac{2m_1}{m_1 + m_2})u_1}$$

Are these equations reasonable?
We do three checks below: pool, baseball, and tennis.

(i) Pool: Cue hits a ball head on => $m_1 = m_2$ => $v_1 = 0$ and $v_2 = u_1$
The cue ball stops and the ball that is hit takes off at $v_2 = u_1$. Looks good!

(ii) Baseball: Toss the ball up and hit it => $m_1 >> m_2$ => $v_1 \approx u_1$ and $v_2 \approx 2u_1$
The massive bat ($m_1 >> m_2$) keeps moving at $v_1 \approx u_1$ and the ball takes off at $v_2 \approx 2u_1$.
We may not be sure of the $v_2 \approx 2u_1$ from intuition, but $v_1 \approx u_1$ is observed as the bat continues to swing. In a second, I will show you intuitively why $v_2 \approx 2u_1$.

(iii) Tennis: Meet the ball with the racket => $m_2 >> m_1$ => $v_1 \approx -u_1$ and $v_2 \approx 0$
The ball reflects back and the racket remains stationary.

Courtesy Daniel Stockman, flickr
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When a ball hits another head on with the same mass, the first ball stops and the second takes off at the speed that the first ball had initially.

This observation is predicted by our formula on the previous page.

Courtesy Erik Drost, flickr
[License: Attribution 2.0 Generic]
Jason Kipnis Home Run (2013)

The case we analyzed on the previous page was tossing the ball up and hitting it. In the photo at the left, the ball is coming towards the batter. However, in both cases, the much more massive bat, compared to the ball, continues on with about the same bat speed. We will see shortly what happens for an incoming pitch.

Courtesy Elizabeth Kuhns, flickr
[Attribution-Noncommercial-NoDerivs 2.0 Generic]
Tennis 2009 Worth

In our tennis example we considered a stationary racket as the ball comes in to hit it. Our physics prediction was that the racket will remain stationary and the ball will bounce off it leaving with the same speed in the opposite direction.

Before we leave this section, let me show you why with the baseball example, the ball leaves at twice the speed of the bat. The main condition here is that the bat is much greater in mass compared to the ball. If we imagine ourselves as a super hero riding along with the bat at speed u along the positive x-direction, we see the ball coming towards us at speed $-u$.

Before, On Ball Field: $u_{bat} = u$ and $u_{ball} = 0$.
Before, Riding on Bat (the primed reference frame): $u'_{bat} = 0$ and $u'_{ball} = -u$.

The secret in going from reference frame of the ball field to bat frame is to subtract u from each speed. To get back to the ball field, we need to add u to each velocity. With the collision, we first check out what happens from the primed frame, i.e., riding with the swinging bat.

After, Riding on Bat (the primed reference frame): $u'_{bat} = 0$ and $u'_{ball} = +u$.
since the ball bounces off the "stationary" bat (for us moving with the bat).

To get back to the ball field reference frame, we use our secret: add u to each velocity.

After, On Ball Field: $u'_{bat} = 0 + u = u$ and $u'_{ball} = +u + u = 2u$.

J2. Elastic Collisions II. Now we treat the general one-dimensional case where two masses are moving and then collide.

The equations are:

conservation of momentum: $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$,

conservation of kinetic energy: $\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$,

where "u" once again stands for an initial velocity and "v" for a final. I continue to use this convention since I hate subscripts and do not want to include "i" for initial and "f" for final in addition to "1" and "2" as subscripts. I remember u is before since u comes before v.

The algebra is going to be messy so I pause for a moment to see if there is a shortcut. Here is what we obtained in the previous section when $u_1 \neq 0$ and $u_2 = 0$.

$$v_1 = (\frac{m_1 - m_2}{m_1 + m_2})u_1 \qquad v_2 = (\frac{2m_1}{m_1 + m_2})u_1$$

If we had the reverse situation, where $u_1 = 0$ and $u_2 \neq 0$ we could get the result by switching the subscripts.

$$v_2 = (\frac{m_2 - m_1}{m_1 + m_2})u_2 \qquad v_1 = (\frac{2m_2}{m_1 + m_2})u_2$$

But you might say that there never will be a collision now because the second mass is going to the right with the left mass at rest. That is correct. A collision would occur only if the velocity of the second mass was negative, i.e., moving to the left initially.

So I boldly claim that the most general case is found by combining the equations for each individual case above.

$$\boxed{v_1 = (\frac{m_1 - m_2}{m_1 + m_2})u_1 + (\frac{2m_2}{m_1 + m_2})u_2}$$

$$\boxed{v_2 = (\frac{2m_1}{m_1 + m_2})u_1 + (\frac{m_2 - m_1}{m_1 + m_2})u_2}$$

Now we will check it out by a brute force calculation to be absolutely sure.

We start with

conservation of momentum: $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$,

conservation of kinetic energy: $\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$.

We have two equations with the two unknowns being the final velocities.

I will use my bracket notation to group the pair of equations.

$$\begin{Bmatrix} m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \\ \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \end{Bmatrix}$$

Multiply the second equation by 2. $\begin{Bmatrix} m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \\ m_1u_1^2 + m_2u_2^2 = m_1v_1^2 + m_2v_2^2 \end{Bmatrix}$

Get mass 1 on the left side. $\begin{Bmatrix} m_1(u_1 - v_1) = m_2(v_2 - u_2) \\ m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2) \end{Bmatrix}$

Factor the second equation. $\begin{Bmatrix} m_1(u_1 - v_1) = m_2(v_2 - u_2) \\ m_1(u_1 + v_1)(u_1 - v_1) = m_2(v_2 + u_2)(v_2 - u_2) \end{Bmatrix}$

Rearrange the second equation. $\begin{Bmatrix} m_1(u_1 - v_1) = m_2(v_2 - u_2) \\ m_1 \dfrac{(u_1 + v_1)}{(v_2 + u_2)}(u_1 - v_1) = m_2(v_2 - u_2) \end{Bmatrix}$

Stare at the equations. We can see $\dfrac{(u_1 + v_1)}{(v_2 + u_2)} = 1$, which leads to $u_1 + v_1 = v_2 + u_2$.

$$\boxed{u_1 - u_2 = v_2 - v_1}$$

We again get the result that

the velocity of approach equals the velocity of separation.

Use $u_1 - u_2 = v_2 - v_1$ in the form $v_2 = u_1 - u_2 + v_1$ with our last expression for equation 1.

$$m_1(u_1 - v_1) = m_2(v_2 - u_2) \quad \Rightarrow \quad m_1(u_1 - v_1) = m_2(u_1 - u_2 + v_1 - u_2)$$

Solve for v_1.

$$m_1u_1 - m_1v_1 = m_2u_1 - m_2u_2 + m_2v_1 - m_2u_2$$

$$m_1u_1 = m_1v_1 + m_2u_1 - m_2u_2 + m_2v_1 - m_2u_2$$

$$m_1u_1 = (m_1 + m_2)v_1 + m_2u_1 - m_2u_2 - m_2u_2$$

$$(m_1 + m_2)v_1 = m_1u_1 - m_2u_1 + m_2u_2 + m_2u_2$$

$$(m_1 + m_2)v_1 = m_1u_1 - m_2u_1 + 2m_2u_2$$
$$(m_1 + m_2)v_1 = (m_1 - m_2)u_1 + 2m_2u_2$$

$$\boxed{v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \left(\frac{2m_2}{m_1 + m_2}\right)u_2}$$

This result is the one we wrote down using symmetry.

To get v_2 the long way, substitute our result for v_1 into $\boxed{u_1 - u_2 = v_2 - v_1}$.

$$u_1 - u_2 = v_2 - \left[(\frac{m_1 - m_2}{m_1 + m_2})u_1 + (\frac{2m_2}{m_1 + m_2})u_2 \right]$$

$$v_2 = u_1 - u_2 + \left[(\frac{m_1 - m_2}{m_1 + m_2})u_1 + (\frac{2m_2}{m_1 + m_2})u_2 \right]$$

$$v_2 = u_1 \left[1 + \frac{m_1 - m_2}{m_1 + m_2} \right] + u_2 \left[-1 + \frac{2m_2}{m_1 + m_2} \right]$$

$$v_2 = u_1 \left[\frac{m_1 + m_2}{m_1 + m_2} + \frac{m_1 - m_2}{m_1 + m_2} \right] + u_2 \left[-\frac{m_1 + m_2}{m_1 + m_2} + \frac{2m_2}{m_1 + m_2} \right]$$

$$v_2 = u_1 \left[\frac{m_1 + m_2 + m_1 - m_2}{m_1 + m_2} \right] + u_2 \left[\frac{-m_1 - m_2 + 2m_2}{m_1 + m_2} \right]$$

$$v_2 = u_1 \left[\frac{2m_1}{m_1 + m_2} \right] + u_2 \left[\frac{m_2 - m_1}{m_1 + m_2} \right]$$

$$\boxed{v_2 = (\frac{2m_1}{m_1 + m_2})u_1 + (\frac{m_2 - m_1}{m_1 + m_2})u_2}$$

Again, we find the result we simply wrote down by way of symmetry earlier.

For a swinging bat $u_1 > 0$ hitting an incoming pitch $u_2 < 0$ where $m_1 >> m_2$ (or $m_2 \approx 0$),

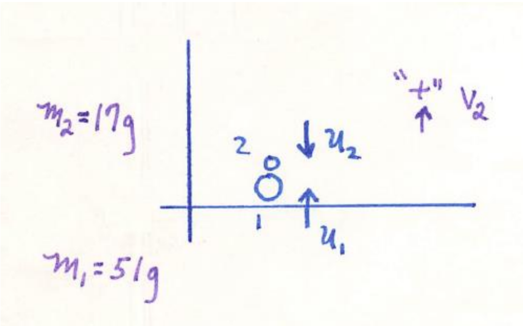
the bat $v_1 = (\frac{m_1 - m_2}{m_1 + m_2})u_1 + (\frac{2m_2}{m_1 + m_2})u_2$ => $v_1 \to u_1$.

The bat maintains its initial velocity once again!

The ball flies off at $v_2 = \left[\frac{2m_1}{m_1 + m_2} \right] u_1 + \left[\frac{m_2 - m_1}{m_1 + m_2} \right] u_2$ => $v_2 = 2u_1 - u_2$. Note $u_2 < 0$.

Therefore, the ball goes flying off at a speed $v_2 > 2u_1$.

J3. The Double Ball Drop.



Many years ago when I taught intro physics, I used to bring two good bouncy rubber balls to class. One was larger than the other. I placed the small one on top of the larger one and carefully dropped them from a height h. When the pair reached the ground, the smaller one shot up and hit the ceiling. I then gathered data to see how high that physics would predict that the smaller ball would reach.

When the lower ball reaches the ground, it bounces and changes direction colliding with the upper ball still on its way down. We have the collision formulas for this one-dimensional type of collision.

$$\boxed{v_1 = (\frac{m_1 - m_2}{m_1 + m_2})u_1 + (\frac{2m_2}{m_1 + m_2})u_2} \qquad \boxed{v_2 = (\frac{2m_1}{m_1 + m_2})u_1 + (\frac{m_2 - m_1}{m_1 + m_2})u_2}$$

I measured the masses of each ball with a laboratory scale and found $m_1 = 51$ g and $m_2 = 17$ g. The heavier ball had three times the mass of the smaller one. So we can write $m_1 = 3m_2$. At the moment of the collision the lower and larger mass is traveling upward with $u_1 > 0$, choosing up as positive. The smaller mass on top is traveling with velocity $u_2 = -u_1$. Why is this? A moment before the large ball makes contact with the floor, it is traveling downward with speed $|u_1|$. It then bounces off the floor and begins traveling upward with speed $|u_1|$. When this occurs, the smaller mass is still traveling downward with speed $|u_1|$. Since $u_1 > 0$, we have $u_2 = -u_1 < 0$ for the smaller ball since down is negative. The collision then occurs. The final velocities are obtained from the formulas. I am really interested in v_2, the velocity of the small ball after the collision.

$$v_2 = (\frac{2m_1}{m_1 + m_2})u_1 + (\frac{m_2 - m_1}{m_1 + m_2})u_2 = (\frac{2 \cdot 3m_2}{3m_2 + m_2})u_1 + (\frac{m_2 - 3m_2}{3m_2 + m_2})(-u_1)$$

$$v_2 = (\frac{6m_2}{4m_2})u_1 + (\frac{-2m_2}{4m_2})(-u_1)$$

$$v_2 = \frac{3}{2}u_1 + \frac{1}{2}u_1$$

$$v_2 = 2u_1$$

How does this translate to the height reached after the bounce?.

When you drop a single ball from height h from rest, we can find the velocity just before hitting the ground from conservation of energy.

$$mgh = \frac{1}{2}mv^2$$

$$h \sim v^2$$

The height is proportional to the square of the velocity. On the bounce, assuming a perfectly elastic bounce, the ball will return to its initial height. Now, since

$$v_2 = 2u_1,$$

squaring both sides gives

$$v_2^2 = 4u_1^2 .$$

The associated heights are then related as

$$h_2 = 4h_1,$$

indicating that the second ball will reach a height four times the height from which I dropped the balls. Remember, dropping from height h_1 gets you u_1 at the ground to get the collision set up in the first place.

Since I dropped the pair from a height of about 1 meter, the little ball is predicted to rise to 4 meters after the collision. The ceiling was about 2.5 meters high so that the little ball should smack into the ceiling. It did and the class was impressed. But sometimes I didn't drop the balls with the little one precisely on top of the bigger one. In those instances, the little ball shot off at an angle. After a few tries, I was able to get it right when I didn't on the first attempt. Of course, the most impressive demonstration is to practice so you get it right the first time and surprise the class with the little ball banging into the ceiling.

Let's see what happens to the larger ball after the collision.

$$v_1 = (\frac{m_1 - m_2}{m_1 + m_2})u_1 + (\frac{2m_2}{m_1 + m_2})u_2 = (\frac{3m_2 - m_2}{3m_2 + m_2})u_1 + (\frac{2m_2}{3m_2 + m_2})(-u_1)$$

$$v_1 = (\frac{2m_2}{4m_2})u_1 + (\frac{2m_2}{4m_2})(-u_1)$$

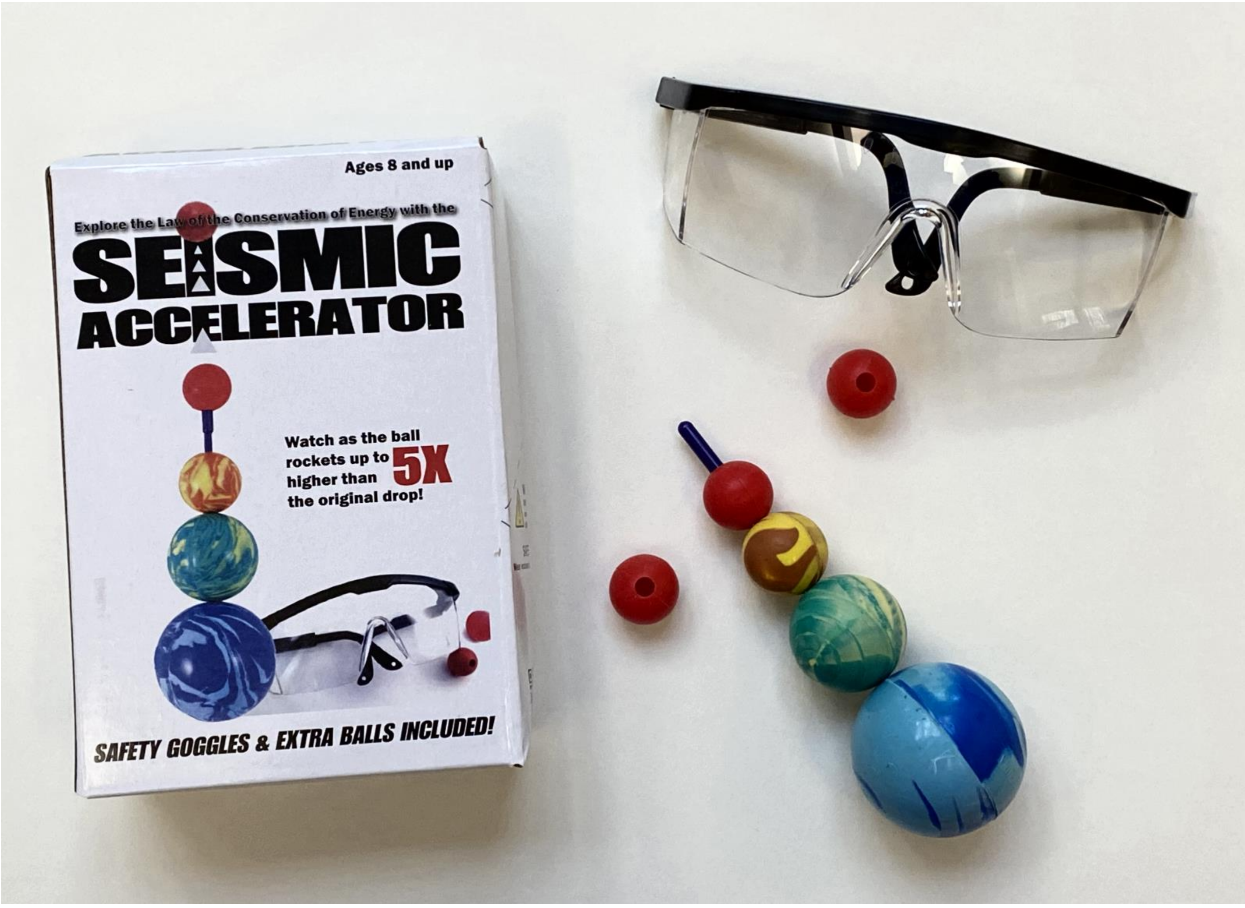
$$v_1 = \frac{1}{2}u_1 + \frac{1}{2}(-u_1)$$

$$v_1 = (\frac{1}{2} - \frac{1}{2})u_1$$

$$v_1 = 0$$

Wow. With the specific mass combination $m_1 = 3m_2$ you get the result that the large ball stays on the ground after the collision! And it did!

Later, toys like the *Seismic Acceleration* and *Astro Blaster* became readily available. They put a few balls with decreasing size on a rod so that you can drop them easily to the ground with them lined up vertically. The extra balls amplifies the effect! But be sure to wear safety glasses using this toy due to the multiplying effect of the collisions. Note that the glasses come with it.



J4. Coefficient of Restitution. If you drop a ball with mass m from height H, the velocity u just before hitting the ground is easily found from conservation of energy.

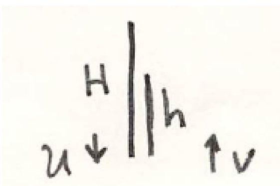
$$K_1 + U_1 = K_2 + U_2$$

$$0 + mgH = \frac{1}{2}mu^2 + 0$$

$$\frac{1}{2}mu^2 = mgH$$

$$u^2 = 2gH$$

$$u = \sqrt{2gH}$$



But after colliding with the floor, the velocity on its way up is $v < u$. We know the bounce is not elastic because the ball will rise to a height $h < H$. We apply conservation of energy again for the way up.

$$K_1 + U_1 = K_2 + U_2$$

$$0 + \frac{1}{2}mv^2 = 0 + mgh$$

$$v^2 = 2gh$$

$$v = \sqrt{2gh}$$

We define the *coefficient of restitution* as below.

$$\boxed{e = \frac{v}{u}}$$

For a collision where two masses are moving, the general formula is given by the ratio of the velocity of separation to the velocity of approach.

$$\boxed{e = \frac{v_2 - v_1}{u_1 - u_2}}$$

For dropping a ball with up being positive, the Earth as mass 1 and ball as mass 2: $u_1 = 0$, $v_1 = 0$, $u_2 = -u$, and $v_2 = v$, we recover our definition for dropping a ball.

$$e = \frac{v_2 - v_1}{u_1 - u_2} = \frac{v - 0}{0 - (-u)} = \frac{v}{u}$$

We can do an experiment to measure the coefficient of restitution for a ball and a specific floor by dropping the ball from height H and see how high h the ball goes up after the bounce. The coefficient of restitution is then determined by

$$e = \frac{v}{u} = \frac{\sqrt{2gh}}{\sqrt{2gH}}$$

$$\boxed{e = \frac{v}{u} = \sqrt{\frac{h}{H}}}$$

Three cases are listed in the table with various descriptive names you might encounter, along with an example of each. The coefficient of restitution depends on both the object dropped and the nature of the floor. A ball bouncing off a rug will not be the same as bouncing off concrete.

Coefficient of Restitution *e*	Description 1	Description 2	Dropping Examples
$e = 1$	Elastic	Perfectly Elastic	Dropping an "Ideal" Superball
$0 < e < 1$	Inelastic	Plastic	Dropping Real Balls
$e = 0$	Perfectly Inelastic	Perfectly Plastic	Dropping Mashed Potatoes

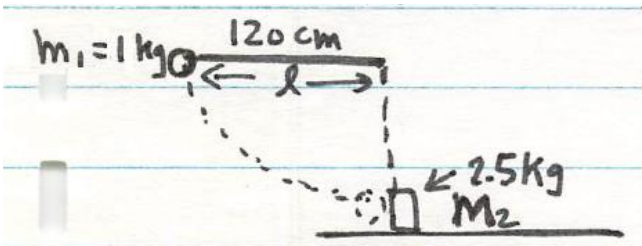
Note that our colliding skaters where they skated together after the collision is an example of a perfectly inelastic collision and the final kinetic energy is not zero as in the mashed potato case above. Remember that we found in an earlier section

$$\frac{K_{after}}{K_{before}} = \frac{m_1}{m_1 + m_2}.$$

Now if the mass you run into is super massive like a wall $m_2 \to \infty$, then you get zero like the dropped mashed potatoes. The lost kinetic energy goes into deforming the mashed potatoes.

Some texts may include explosions, calling those super elastic since kinetic energy is gained. But I'd rather not mix explosions with the regular types of collisions we are working with.

J5. The Pendulum Hitting a Block.



A mass $m_1 = 1.0 \text{ kg}$ is released from rest, swinging down from 90.0° on a pendulum of length $l = 120.0 \text{ cm}$ and eventually collides with a mass $m_2 = 2.5 \text{ kg}$ at rest on the floor. Find the final velocities of the masses for an elastic collision and no friction between m_2 and the ground.

The equations for the final velocities derived in class for a collision where mass m_1 is moving in the x-direction towards m_2 at rest are

$$v_1 = (\frac{m_1 - m_2}{m_1 + m_2})u_1 \quad \text{and} \quad v_2 = (\frac{2m_1}{m_1 + m_2})u_1 .$$

We do not have to rederive these just like we never rederived our kinematics formulas when we used them in problems. The velocity equations do not need to be memorized. However, you should have the four basic kinematic equations memorized.

What is missing so far is u_1. How do we get u_1? The answer is conservation of energy.

$$m_1 g l = \frac{1}{2} m_1 u_1^2 \quad => \quad 2gl = u_1^2 \quad => \quad u_1 = \sqrt{2gl} = \sqrt{2(9.8 \frac{\text{m}}{\text{s}^2})(1.20 \text{ m})} = \sqrt{23.52} \frac{\text{m}}{\text{s}} = 4.850 \frac{\text{m}}{\text{s}}$$

We can now find the final velocities.

$$v_1 = (\frac{m_1 - m_2}{m_1 + m_2})u_1 = (\frac{m_1 - m_2}{m_1 + m_2})\sqrt{2gl} = (\frac{1.0 \text{ kg} - 2.5 \text{ kg}}{1.0 \text{ kg} + 2.5 \text{ kg}}) \cdot 4.850 \frac{\text{m}}{\text{s}} = (\frac{-1.5}{3.5}) \cdot 4.850 \frac{\text{m}}{\text{s}}$$

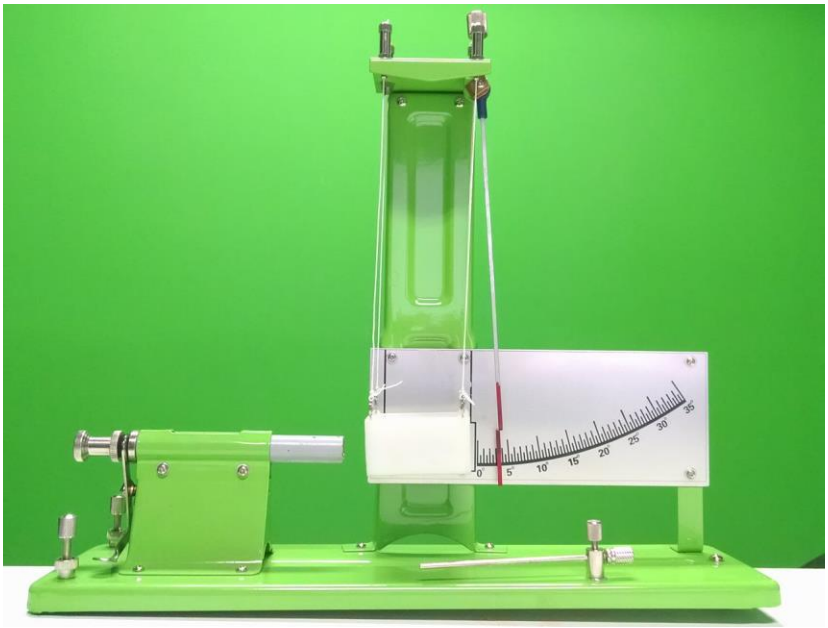
$$v_1 = -\frac{3}{7} \cdot 4.850 \frac{\text{m}}{\text{s}} = -2.1 \frac{\text{m}}{\text{s}}$$

$$v_2 = (\frac{2m_1}{m_1 + m_2})\sqrt{2gl} = (\frac{2 \cdot 1.0 \text{ kg}}{1.0 \text{ kg} + 2.5 \text{ kg}}) \cdot 4.850 \frac{\text{m}}{\text{s}} = (\frac{2.0}{3.5}) \cdot 4.850 \frac{\text{m}}{\text{s}}$$

$$v_1 = \frac{4}{7} \cdot 4.850 \frac{\text{m}}{\text{s}} = 2.8 \frac{\text{m}}{\text{s}}$$

The pendulum mass m_1 by the way moves back to the left after impact since $v_1 < 0$.

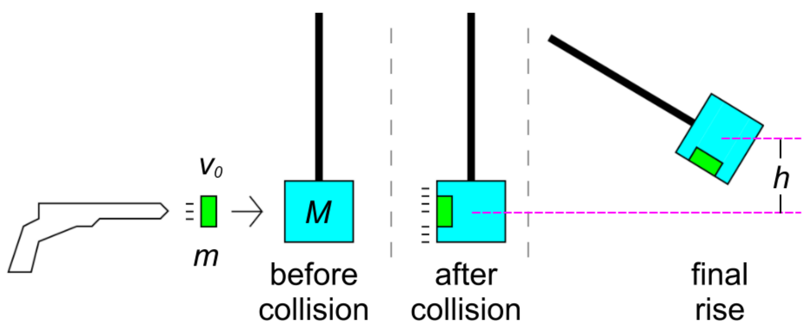
J6. The Ballistic Pendulum. In the old days they used collision physics to measure the speed of a bullet. A bullet with mass m was fired into a block of wood having mass M which was attached to a pendulum. The combination then rose up to a height h.



Courtesy Steven H. Keys
http://www.keysphotography.com
Wikipedia, License: Creative Commons Attribution 4.0 Intl.

A student ballistic pendulum appears in the left photo. There are two ropes attached to the block. Decades ago we had a lab with the ballistic pendulum to reinforce the formulas we were learning in class.

A simplified schematic of the ballistic pendulum appears next.



Adapted from Burn, who adapted from Makeemlighter Wikipedia, Public Domain

There is a story on the Internet where a former student says her professor used to do the demonstration live with a rifle in the tiered 300-seat auditorium in Herzstein Hall, Rice University, Houston, Texas, USA.

Problem. Find the bullet speed in m/s for a 12-g bullet fired into a 7-kg block of wood where the combination rises 5 cm. Data from Numerade.

Solution. Let's start with conservation of momentum before and after the collision.

$$mv_0 = (m+M)v ,$$

where v_0 is the speed of the bullet and v is the speed of the bullet-wood combination after the collision. The mass of the bullet is m and the mass of the block is M. For the second phase, where the bullet-wood combination rises, we can use conservation of energy.

$$\frac{1}{2}(m+M)v^2 = (m+M)gh \quad => \quad v = \sqrt{2gh}\,.$$

We substitute this v into the momentum equation $mv_0 = (m+M)v$.

$$mv_0 = (m+M)\sqrt{2gh}$$

$$v_0 = \left(\frac{m+M}{m}\right)\sqrt{2gh}$$

$$\boxed{v_0 = \left(1+\frac{M}{m}\right)\sqrt{2gh}}$$

$$v_0 = \left(1+\frac{7000\text{ g}}{12\text{ g}}\right)\sqrt{2\left(9.8\,\frac{\text{m}}{\text{s}^2}\right)(0.05\text{ m})}$$

$$v_0 = (584.33)\sqrt{0.9800}\,\frac{\text{m}}{\text{s}}$$

$$v_0 = 578.46\,\frac{\text{m}}{\text{s}}$$

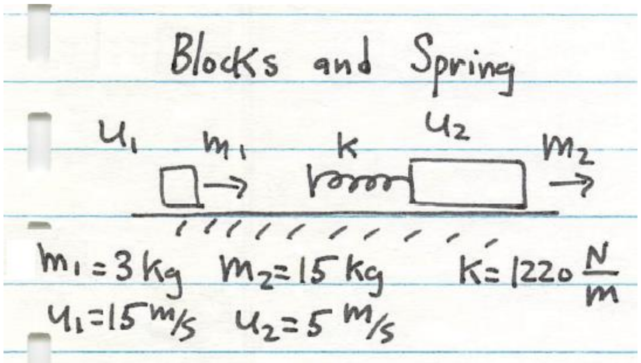
$$v_0 = 578.46\,\frac{\text{m}}{\text{s}}\left(\frac{1\text{ km}}{1000\text{ m}}\right)\left(\frac{3600\text{ s}}{1\text{ h}}\right) = 2082\,\frac{\text{km}}{\text{h}}$$

$$v_0 = 2082\,\frac{\text{km}}{\text{h}}\left(\frac{1\text{ mi}}{1.609\text{ km}}\right) = 1294\,\frac{\text{mi}}{\text{h}}$$

$$\boxed{v_0 = 580\,\frac{\text{m}}{\text{s}}} \qquad \boxed{v_0 = 2100\,\frac{\text{km}}{\text{h}}} \qquad \boxed{v_0 = 1300\,\frac{\text{mi}}{\text{h}}}$$

The speed of sound is 340 m/s. This bullet travels faster than sound speed and therefore, there will be a sonic boom.

J7. Blocks and a Spring.



Problem. A 3-kg mass is traveling to the right at 15 m/s. It reaches a 15-kg block that is traveling slower at 5 m/s to the right. As the 3-kg mass catches up to the 15-kg mass, it begins to compress the spring which has spring constant 1220.0 N/m. Find the maximum compression of the spring and velocity of the system at maximum compression. The collision is elastic and there is no friction.

Solution. Elastic and no friction mean we can use conservation of energy with kinetic and potential energies:

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}kx^2,$$

where x is maximum compression with $v_1 = v_2 = v$, the velocity when the two mass move together at the same speed at the moment of maximum compression. Therefore,

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}(m_1 + m_2)v^2 + \frac{1}{2}kx^2.$$

Conservation of momentum gives

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \quad => \quad m_1u_1 + m_2u_2 = (m_1 + m_2)v \quad => \quad v = \frac{m_1u_1 + m_2u_2}{m_1 + m_2}.$$

We will now enter numbers into our two main equations:

$$v = \frac{m_1u_1 + m_2u_2}{m_1 + m_2} \quad \text{and} \quad \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}(m_1 + m_2)v^2 + \frac{1}{2}kx^2.$$

$$v = \frac{(3)(15) + (15)(5)}{3 + 15} = \frac{45 + 75}{18} = 6.667\,\frac{\text{m}}{\text{s}}$$

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}(m_1 + m_2)v^2 + \frac{1}{2}kx^2 \quad => \quad m_1u_1^2 + m_2u_2^2 = (m_1 + m_2)v^2 + kx^2$$

$$(3)(15)^2 + (15)(5)^2 = (3 + 15)(6.667)^2 + (1220.0)x^2$$

$$675 + 375 = 800.08 + 1220.0x^2$$

Normally I would keep 800.08 and round off last, but I am going for only 2 significant figures.

$$1050 = 800 + 1220x^2 \quad => \quad 250 = 1220x^2$$

$$x^2 = \frac{250}{1220} = 0.205$$

$$x = \sqrt{0.205}$$

$$\boxed{x = 0.45 \text{ m}}$$

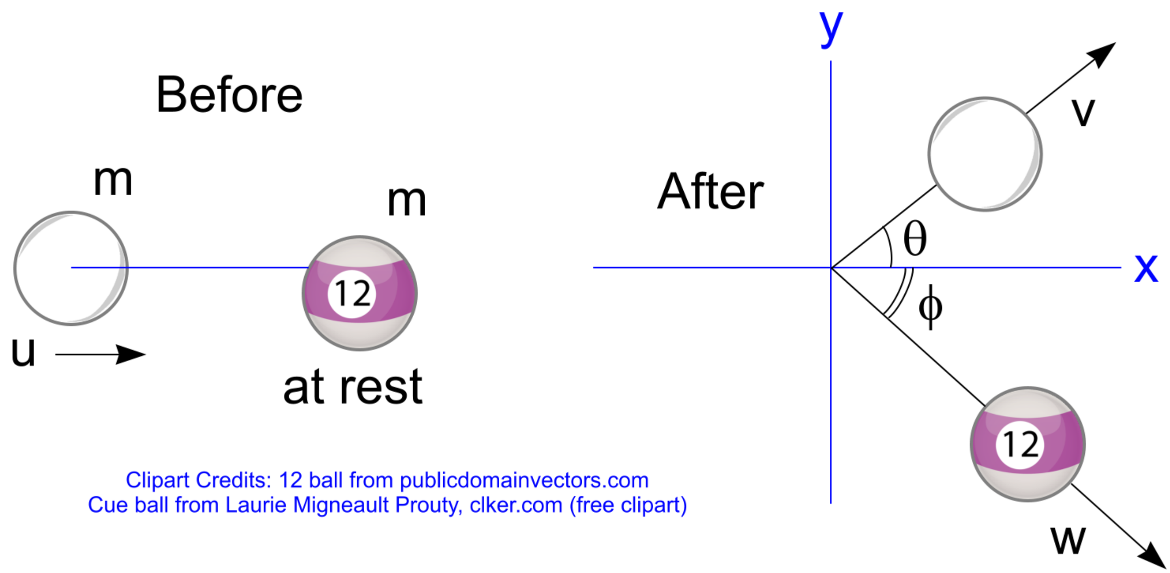
or

$$\boxed{x = 45 \text{ cm}}$$

$$v = 6.667 \frac{\text{m}}{\text{s}}$$

$$\boxed{v = 6.7 \frac{\text{m}}{\text{s}}}$$

J8. Billiards: Collisions in Two Dimensions. We apply our laws of physics to billiards. In the typical shot one ball is sent to collide with another. See the figure below.



The cue ball (white) hits the 12 ball a little off center. The collision results in the balls going off at an angle after the collision. The masses of the balls are the same. Let's analyze this problem and see what we can discover. The initial speed of the cue ball is u and it travels towards the stationary 12 ball. The speeds after the collision are v and w, as my preference is not to use subscripts. We start with conservation of momentum. We have two dimensions to consider: x and y.

$$mu = mv\cos\theta + mw\cos\phi$$

$$0 = mv\sin\theta - mw\sin\phi$$

For an elastic collision we add conservation of energy.

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + \frac{1}{2}mw^2$$

We can write these three equations as follows.

$$\left\{\begin{array}{l} mu = mv\cos\theta + mw\cos\phi \\ mv\sin\theta = mw\sin\phi \\ mu^2 = mv^2 + mw^2 \end{array}\right\}$$

The mass factors divide out everywhere.

$$\begin{Bmatrix} u = v\cos\theta + w\cos\phi \\ v\sin\theta = w\sin\phi \\ u^2 = v^2 + w^2 \end{Bmatrix}$$

We have three equations but four unknowns: v, θ, w, and ϕ, given u. The third equation, $u^2 = v^2 + w^2$, looks like a Pythagorean equation! It means that the outgoing balls are perpendicular to each other! Here is another way to see it. Write the conservation of momentum as a vector equation.

$$m\vec{u} = m\vec{v} + m\vec{w}$$

The masses cancel out.

$$\vec{u} = \vec{v} + \vec{w}$$

Now we can use the dot product.

$$u^2 = \vec{u}\cdot\vec{u} = (\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w})$$

$$u^2 = \vec{v}\cdot\vec{v} + \vec{w}\cdot\vec{v} + \vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{w} \quad => \quad u^2 = \vec{v}\cdot\vec{v} + 2\vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{w}$$

$$u^2 = v^2 + 2\vec{v}\cdot\vec{w} + w^2$$

But we know $u^2 = v^2 + w^2$ from our third equation of our three earlier equations.

Therefore, $u^2 = v^2 + w^2$ means $\vec{v}\cdot\vec{w} = 0$, i.e., $\theta + \phi = 90°$.

Two cases are shown below, with two different offsets for the points of collision contact. Remember, our model assumes elastic collisions. Our result is a profound general property of pool shots. In our YouTube video we will include a some pool collisions so you can judge for yourself concerning the 90°.