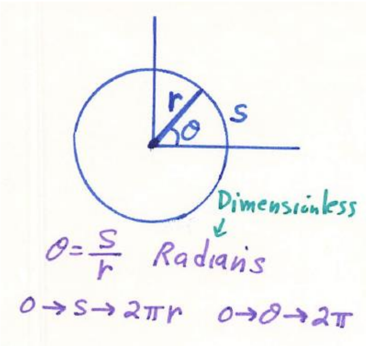
K0. Radians. Angle measure is fundamental in rotational motion.



We have been using degrees for angle measurements so far in our course. We will also find it convenient to use radians. An angle measure in radians is defined as

$$\theta = \frac{s}{r},$$

where s is the arc length with radius r spanned by the angle.

The radian measure is dimensionless since distance units cancel out when the ratio of arc length and radius is taken. But we handle radian measure as if the angle had a dimension by often appending radians to the number. When the arc length goes from 0 to $2\pi r$, the angle measure in radians goes from 0 to 2π.

$$0 \to s \to 2\pi r$$

$$0 \to \theta \to 2\pi$$

To convert between degrees and radian we note

$$180° = \pi \text{ radians},$$

where we can abbreviate radians as rad,

$$180° = \pi \text{ rad}.$$

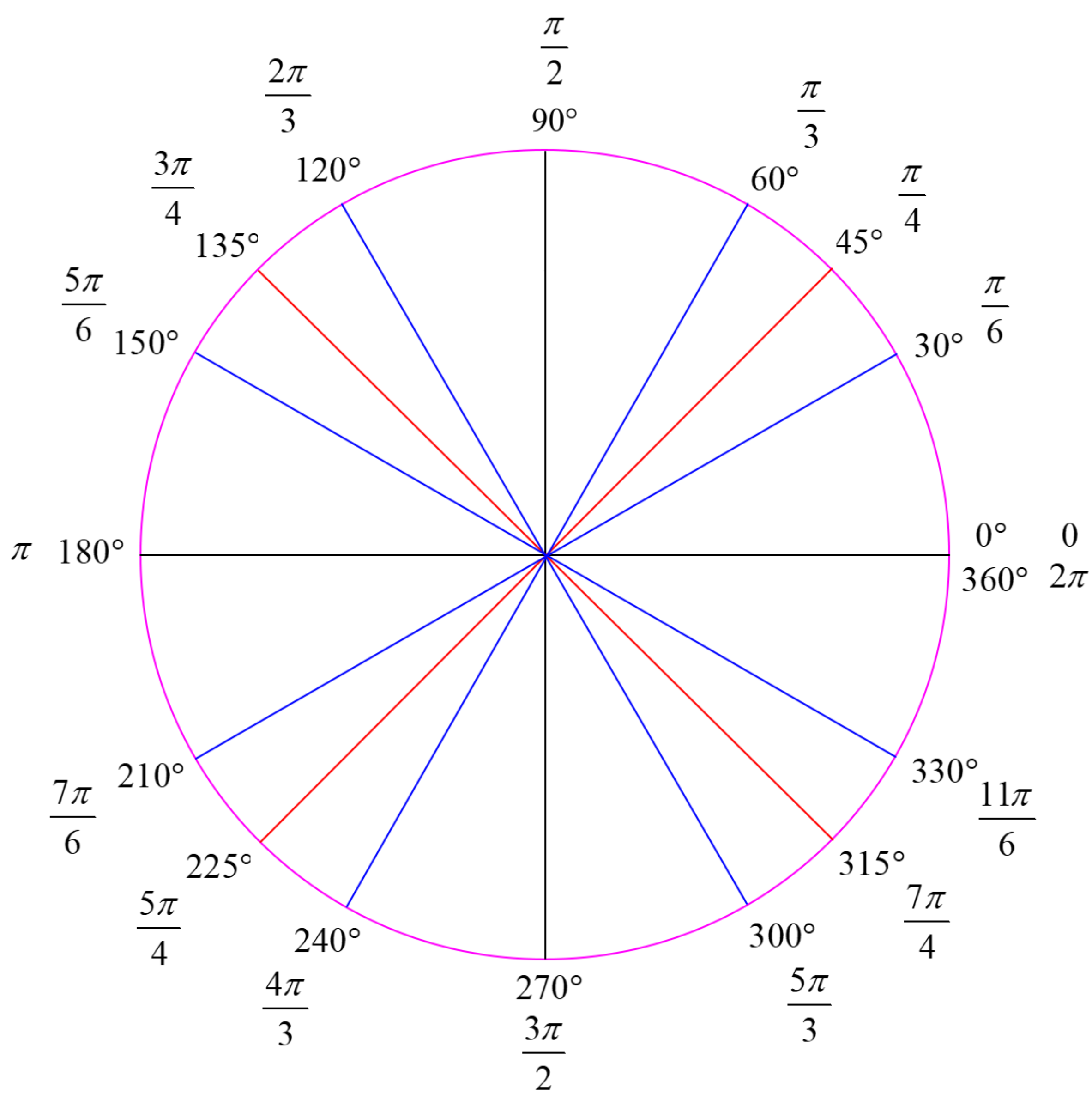
We can also write $180° = 3.14159\ldots \text{ rad}$

To convert $\theta = 45°$ to radians, do the usual conversion of units trick:

$$\theta = 45° = 45° \cdot \frac{\pi \text{ rad}}{180°} = \frac{\pi}{4}\text{rad} = \frac{3.14159\ldots}{4}\text{rad} = 0.7854 \text{ rad}$$ to 4 significant figures.

You can also leave it as $\theta = \frac{\pi}{4}\text{rad}$ or even $\theta = \frac{\pi}{4}$ as the π implies that we mean radians.

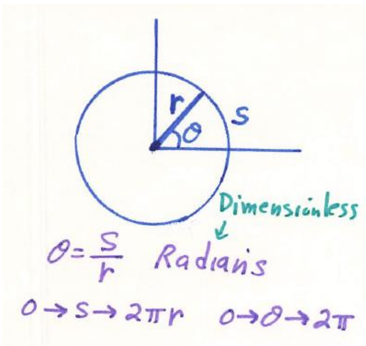
Radians are introduced in trigonometry. As a review, see the figure on the next page that gives some common angles in both degrees and radians.



K1. Rotational Kinematics. The subject of rotational kinematics can be approached by analogy with translational kinematics. We start with the physical quantities.

Translational Physical Quantities				Rotational Physical Quantities			
Quantity	Sym-bol	Definition	Units	Quantity	Sym-bol	Definition	Units
Position	x	Measure from a reference.	m	Angle	θ	Measure from a reference.	rad
Velocity	v	$\frac{dx}{dt} = \lim_{t \to 0} \frac{\Delta x}{\Delta t}$	$\frac{\text{m}}{\text{s}}$	Angular Velocity	ω	$\frac{d\theta}{dt} = \lim_{t \to 0} \frac{\Delta \theta}{\Delta t}$	$\frac{\text{rad}}{\text{s}}$
Acceleration	a	$\frac{dv}{dt} = \lim_{t \to 0} \frac{\Delta v}{\Delta t}$	$\frac{\text{m}}{\text{s}^2}$	Angular Acceleration	α	$\frac{d\omega}{dt} = \lim_{t \to 0} \frac{\Delta \omega}{\Delta t}$	$\frac{\text{rad}}{\text{s}^2}$

The physical quantities angular velocity and angular acceleration arise naturally. Note that the angular acceleration is not to be confused with the centripetal acceleration of circular motion $a_c = \frac{v^2}{r}$, where v is velocity along the circle and r is the radius of the circle. However, this velocity along the circle is related to the angular velocity. Think radial for a_c and tangent for α.



The key is to use $\theta = \frac{s}{r}$. We arrange this relation as

$$s = r\theta$$

and consider a delta angle for a fixed radius,

$$\Delta s = r\Delta\theta \, .$$

Then

$$\frac{\Delta s}{\Delta t} = r\frac{\Delta\theta}{\Delta t}$$

and

$$\lim_{t\to 0}\frac{\Delta s}{\Delta t} = r\lim_{t\to 0}\frac{\Delta\theta}{\Delta t} \, ,$$

$$\frac{ds}{dt} = r\frac{d\theta}{dt}$$

$$v = r\omega \, .$$

If we keep going,

$$\lim_{t\to 0}\frac{\Delta v}{\Delta t} = r\lim_{t\to 0}\frac{\Delta\omega}{\Delta t} \, ,$$

$$\frac{dv}{dt} = r\frac{d\omega}{dt} \, ,$$

$$a = r\alpha \, .$$

I often like to write these two connections with the angular quantities first on the right side:

$$\boxed{v = \omega r} \quad \text{and} \quad \boxed{a = \alpha r}.$$

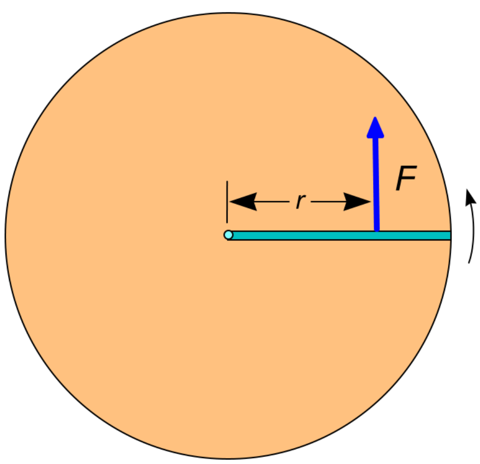
Then, when I read these equations out loud, they have a nice ring to them. Now we are ready for the rotational kinematic equations, again by analogy.

$$x \to \theta \qquad v \to \omega \qquad a \to \alpha$$

Translational Kinematics	Rotational Kinematics
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x = x_0 + v_0 t + \frac{1}{2}at^2$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
$x = x_0 + \frac{1}{2}(v_0 + v)t$	$\theta = \theta_0 + \frac{1}{2}(\omega_0 + \omega)t$
$2a(x - x_0) = v^2 - v_0^2$	$2\alpha(\theta - \theta_0) = \omega^2 - \omega_0^2$

K2. Rotational Dynamics. This section introduces rotational dynamics. Later we will dedicate an entire chapter to the subject. What do we have to do to get rotational motion?

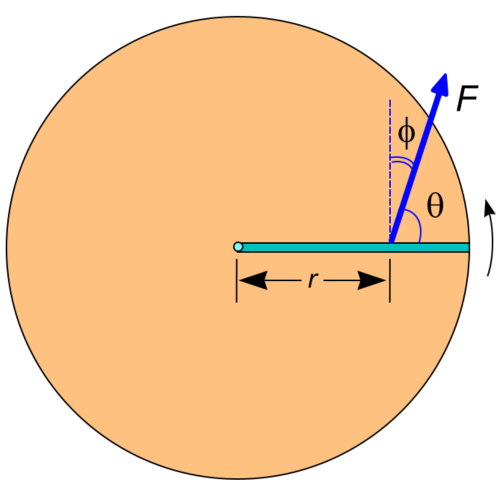
1) Torque (Analogous to Force)



We apply a force as shown in the figure and notice that the distance r from the center matters. If the force is applied farther away the center, its effect on the rotation is greater.

We call this "twisting force" that depends on both r and F the *torque* and define torque in this case as the product of the two. Note that this formula is a definition. There is no law of physics, yet.

$$\tau = rF$$



What about if the force is at an angle as shown in the second figure? In that case, we want the force component perpendicular to the radius.

$$\tau = rF\cos\phi = rF\sin\theta$$

The direction of the twisting effect or torque is "counterclockwise." But the circle spins! So here is the trick we use to describe the direction. You curve your right hand along the counterclockwise direction and note where you thumb points. The thumb is used to define a unique direction. It is out of the page, perpendicular to the disk.

Right-Hand Rule for Rotation.
Public Domain, Wikipedia.
Credit Line: Schorschi2, Wizard191.

The right-hand rule for rotation is shown at the left. The curved fingers follow the circular motion. The thumb gives the direction. In this way, we can promote our scalar version of torque to a vector. We write the torque now as

$$\vec{\tau} = rF\sin\theta\, n,$$

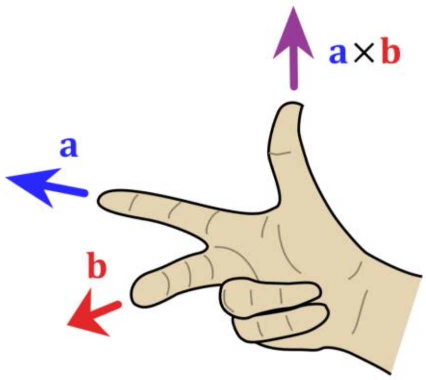
where n is a unit vector pointing in the direction of the thumb. We can promote r to a vector $\vec{r}$. Since force is a vector $\vec{F}$, we have two vectors that serve as input in determining a third vector $\vec{\tau}$. The convention is to use the following vector notation to mean the above expression

$$\boxed{\vec{\tau} = \vec{r}\times\vec{F} = rF\sin\theta\, n}.$$

This manipulation where two vectors are combined in some way to get a third vector is called the *vector cross product* or simply *cross product*. In mathematical terms, the cross product of two vectors can be written in general as

$$\vec{a}\times\vec{b} = ab\sin\theta\, n,$$

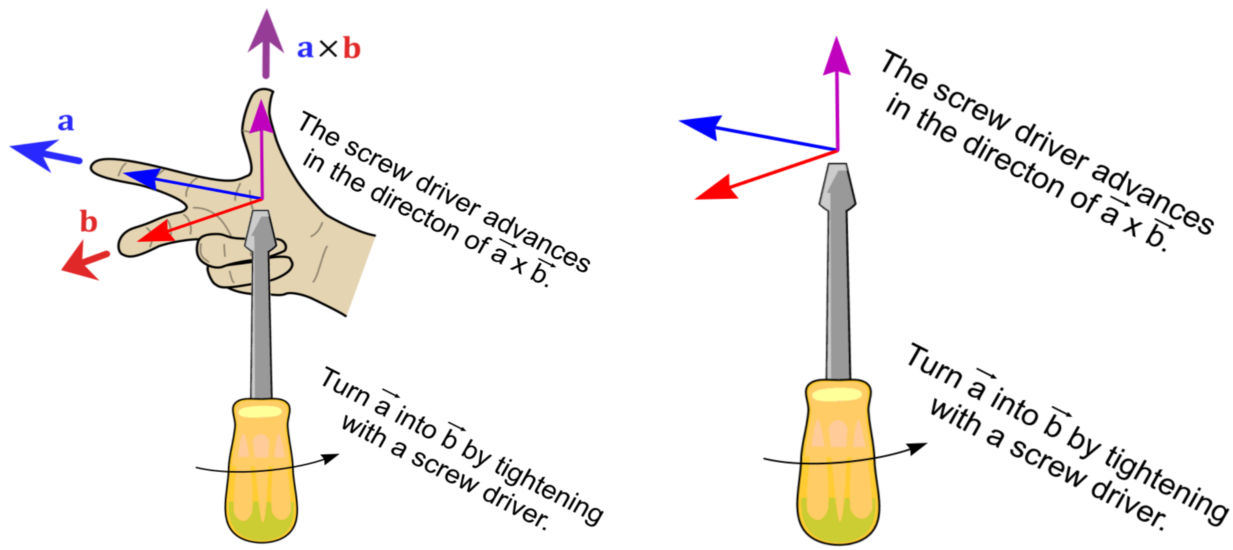
where the following variation of the right-hand rule can be used for the direction.



Right-Hand Rule for the Cross Product.
Courtesy Acdx, Wikimedia
GNU Free Documentation License

I personally do not like the hand rule since I learned about cross products thinking about a screw driver. So I prefer the "screw driver" rule. In the USA, when you tighten a screw you place the screw driver on the screw and rotate clockwise. There is a very old saying "Righty-Tighty and Lefty-Loosey" to get the turn correctly on this standard screw thread. Turning to the right is clockwise for tightening and turning to the left is counterclockwise for loosening. We are assuming right-handed threads here for the screws. There are some instances where left-handed threads are used. See the next figure for the "screw driver" method. You pick your favorite way. In the figure below, the screw driver has to approach from below and it points upward.

Screw Driver Rule for the Cross Product



Hand Courtesy Acdx (Wikimedia), Screw Driver Courtesy openclipart.org

For $\vec{b}\times\vec{a}$ we need to turn the red vector into the blue one in the above figure and the screw driver must then face downward. The result is in the opposite direction: $\vec{b}\times\vec{a} = ab\sin\theta\,(-n)$. Therefore,

$$\vec{b}\times\vec{a} = -\vec{a}\times\vec{b}$$

Note that the cross product of any vector with itself is zero since the angle between them is zero and the sine of zero is zero:

$$\vec{A}\times\vec{A} = A^2(\sin 0°)n = 0\,.$$

$$\boxed{\vec{A}\times\vec{A} = 0}$$

2) Angular Momentum (Analogous to Linear Momentum). What is the analogous version of $\vec{F} = \frac{d\vec{p}}{dt}$ for rotation? In other words, we need the analogous dynamical equation for rotation. The important equation we have so far is a definition: $\vec{\tau} = \vec{r}\times\vec{F}$. So that is where we will start.

$$\vec{\tau} = \vec{r}\times\vec{F} \quad => \quad \vec{\tau} = \vec{r}\times\frac{d\vec{p}}{dt}$$

Now we have introduced physics via Newton's Second Law: $\vec{F} = \dfrac{d\vec{p}}{dt}$.

At this stage we note the following trick: $\dfrac{d\vec{r}}{dt} \times \vec{p} = 0$? Here is why this cool relation is true:

$$\frac{\vec{dr}}{dt} \times \vec{p} = \vec{v} \times \vec{p} = \vec{v} \times m\vec{v} = m\vec{v} \times \vec{v} = 0$$ since a cross product of a vector with itself is 0.

Because of this trick we can do the following.

$$\vec{\tau} = \vec{r} \times \vec{F} \quad \Rightarrow \quad \vec{\tau} = \vec{r} \times \frac{d\vec{p}}{dt} \quad \Rightarrow \quad \vec{\tau} = 0 + \vec{r} \times \frac{d\vec{p}}{dt} \quad \Rightarrow \quad \vec{\tau} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$\vec{\tau} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

The last step involves the product rule for derivatives used in reverse.

By analogy, compare $\vec{F} = \dfrac{d\vec{p}}{dt}$ with $\vec{\tau} = \dfrac{d}{dt}(\vec{r} \times \vec{p})$.

The torque $\vec{\tau} = \dfrac{d}{dt}(\vec{r} \times \vec{p})$ is analogous to force $\vec{F} = \dfrac{d\vec{p}}{dt}$.

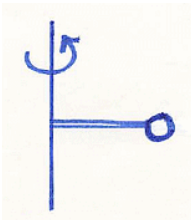
Therefore, $\vec{r} \times \vec{p}$ is analogous to linear momentum.

We call this rotational momentum by the name *angular momentum*.

The letter L is used for angular momentum. You might see lower case in some texts.

We now have our rotational dynamic equations:

$$\boxed{\vec{L} = \vec{r} \times \vec{p}} \quad \text{and} \quad \boxed{\vec{\tau} = \frac{d\vec{L}}{dt}}.$$



3) Rotational Inertia. Next, we would like to investigate the analog of mass. Imagine a ball with mass m attached to a very thin rod of length r that can spin freely about the vertical axis with no friction. We apply a torque on the ball to get the ball to go in a circular path, where the radius is r. Let the velocity of the ball be v as the ball moves along its circular path.

The kinetic energy is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(\omega r)^2 = \frac{1}{2}m\omega^2 r^2 = \frac{1}{2}mr^2\omega^2 = \frac{1}{2}I\omega^2,$$

where the rotational inertia, by analogy, is

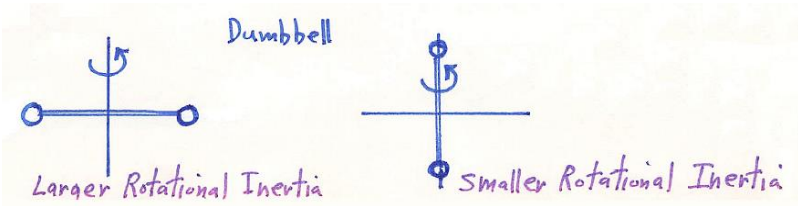
$$\boxed{I = mr^2}.$$

This rotational inertia is called the *moment of inertia*. For a group of particles rotating together with angular velocity ω, the kinetic energy is

$$K = \sum_i \frac{1}{2}m_i v_i^2 = \sum_i \frac{1}{2}m_i(\omega r_i)^2 = \frac{1}{2}\sum_i (m_i r_i^2)\omega^2 = \frac{1}{2}I\omega^2,$$

where the moment of inertia is now $\boxed{I = \sum_i m_i r_i^2}$

The farther out a mass is, the greater contribution to the total moment of inertia.



The orientation on the left has the larger moment of inertia.

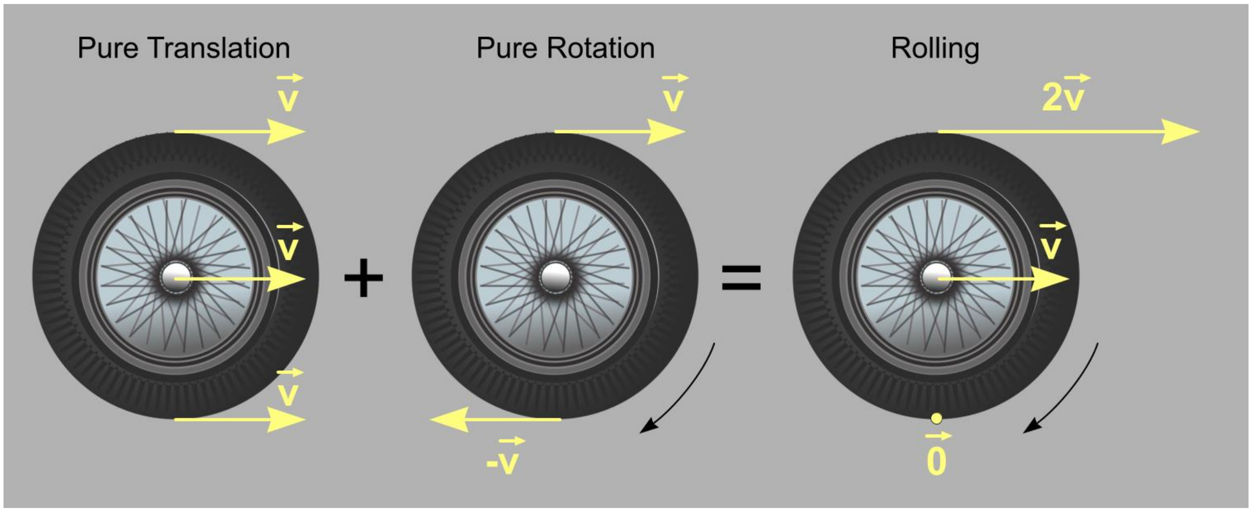
One last comment for this section involves the kinetic energy for rotation. If you have a rolling ball of mass M, then you have rotation, but also translational motion as the ball rolls across the room with some linear velocity v. The total kinetic energy then consists of two parts: the translational kinetic energy and the rotational kinetic energy.

$$K = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

If there is no slipping as the ball rolls, then how is the velocity v related to the angular velocity ω? In order to answer this question, first refer to the next figure which indicates how you can add pure translation to purely circular motion in order to get rolling motion. For the middle wheel, we write

$$v = \omega r,$$

where the r is the radius of the wheel. Note the points where the speed is 2v and zero.



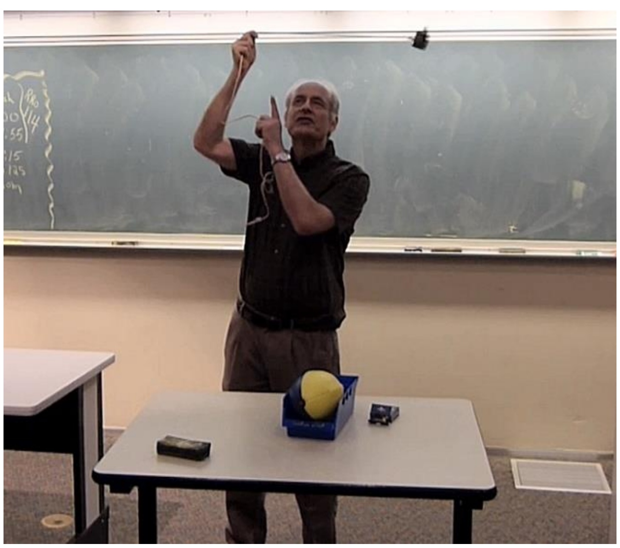
Wheel Clipart Courtesy openclipart.org

The total kinetic energy $K = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$ is then

$$K = \frac{1}{2}Mv^2 + \frac{1}{2}I(\frac{v}{r})^2 .$$

We will come back to this equation later, after a chapter dedicated to moment of inertia I .

K3. Centripetal and Tangential Acceleration. In Chapter E we encountered twirling a mass on a string. At that time we analyzed circular motion with constant velocity, a situation after I got the mass up to speed twirling it. Now we consider the angular acceleration as the mass gets up to speed.

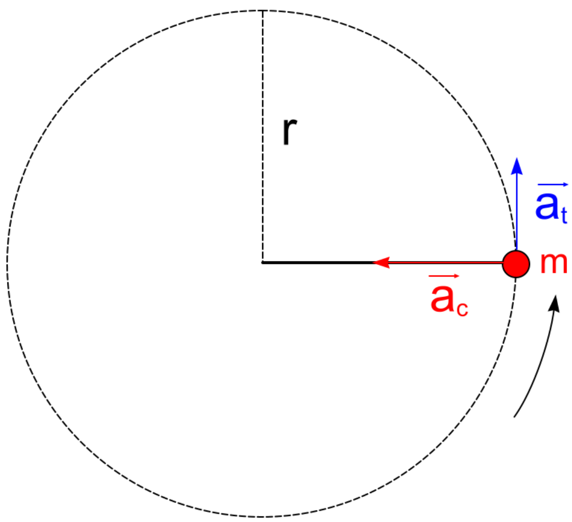


There are two types of acceleration to consider. These accelerations are listed below with descriptions.

1. The Centripetal Acceleration $a_c = \frac{v^2}{r}$ which points inward to the center of the circular path. This acceleration depends on the velocity v and can also be called the radial acceleration $a_r = a_c$.

2. The Tangential Acceleration $a_t = \alpha r$ due to the angular acceleration α .

Since acceleration is a vector, we can assign the directions for the centripetal and tangential accelerations. The directions are shown in the figure, which can be considered a top view of the twirling mass.

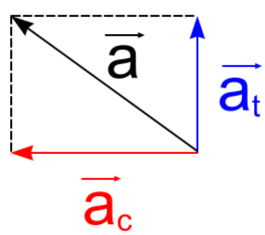


Note the important following fact. You can write the angular acceleration as a vector $\vec{\alpha}$ using the right-hand rule for the direction. The magnitude is related to the tangential acceleration shown in the figure as

$$a_t = \alpha r \; .$$

But the vector direction for a_t does NOT involve any right-hand rule. The tangential acceleration points along the tangent to the circle at each point.

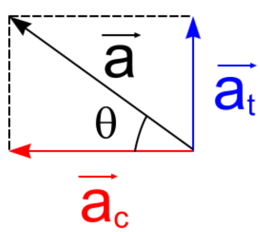
The total acceleration vector, the resultant, is given by adding the tangential and acceleration vectors.



The resultant is shown in the figure and given as an equation below.

$$\vec{a} = \vec{a_c} + \vec{a_t}$$

The direction will change from point to point as the direction of $\vec{a_t}$ changes both in direction and magnitude since $a_t = \alpha r$.



The angle shown in the left figure can be calculated from the magnitudes of the acceleration vector components:

$$\tan\theta = \frac{a_t}{a_c} \; .$$

Let's get some real-world data and formulate a problem. A racing car comes to mind, one taking a curve at high speed and accelerating.

Accept as given data that a race car is taking a 50.0-meter radius curve at 175 km/h with the car accelerating forward at 20.0 km/h per second.

(i) Find the g-force due to the circular motion. In which direction does the driver experience this g-force?

(ii) Find the g-force due to the forward acceleration. In which direction does the driver experience this g-force?

(iii) Finally Then find the total g-force experienced by the driver. In which direction does the driver experience this resultant g-force? Take g = 9.81 m/s^2.

British Touring Car Championship (BTCC), Silverstone Circuit
Photo Courtesy Rachel Clarke, flickr. Photo taken on August 22, 2010,
Northamptonshire, United Kingdom. License: Attribution-NonCommercial 2.0 Generic

Data Given in Problem: $r = 50.0\text{ m}$, $v = 175\ \frac{\text{km}}{\text{h}}$, $a_t = 20.0\ \frac{\text{km}}{\text{h}\cdot\text{s}}$, and $g = 9.81\ \frac{\text{m}}{\text{s}^2}$. Note that everything is listed to 3 significant figures. So I will report my final answers to 3 significant figures. Since I will be using $g = 9.81\ \frac{\text{m}}{\text{s}^2}$ at some point, I would like to have everything in meters and seconds.

$$v = 175\ \frac{\text{km}}{\text{h}} = 175 \cdot \frac{1000.0\text{ m}}{3600.0\text{ s}} = 48.6111 \frac{\text{m}}{\text{s}}$$

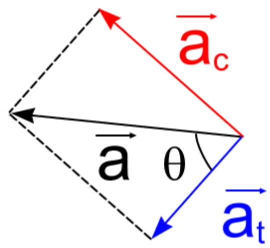
$$a_t = 20.0\ \frac{\text{km}}{\text{h}\cdot\text{s}} = 20.0 \cdot \frac{1000.0\text{ m}}{3600.0\text{ s}} \cdot \frac{1}{\text{s}} = 5.55556 \frac{\text{m}}{\text{s}^2}$$

I always keep extra significant figures and round off last.

(i) g-force for centripetal motion: $\frac{a_c}{g} = \frac{1}{g}\frac{v^2}{r} = \frac{1}{9.81}\frac{(48.6111)^2}{50.0} = 4.8176$. Therefore $\boxed{a_c = 4.82g}$.
The acceleration is towards the center of the circular path. In the photo, the driver will feel the door of the racing car that we can see in the photo pushing on them towards the grass. The driver will experience 4.82 times their weight in that direction.

(ii) g-force for the forward acceleration: $\frac{a_t}{g} = \frac{5.55556}{9.81} = 0.56632$. Therefore $\boxed{a_t = 0.566g}$. The acceleration is forward. The driver will feel the back part of the seat pushing forward on their back. The driver will experience 0.566 times their weight in the forward direction.

(iii) Total g-force.



Since $a_c >> a_t$, we expect a large angle for the angle shown in the figure.

$$a = \sqrt{a_c^2 + a_t^2} = \sqrt{4.8176^2 + 0.56632^2} = 4.85077 \quad => \quad \boxed{a = 4.85g}$$

$$\tan\theta = \frac{a_c}{a_t} \quad => \quad \theta = \tan^{-1}\frac{a_c}{a_t} = \tan^{-1}\frac{4.8176}{0.56632} \quad => \quad \boxed{\theta = 83.3°}$$

The direction of the resultant acceleration is 83.3° from the forward direction towards the driver's right. It is very close to the direction in which the center of the circular path lies. The center of the circle is 90° to the right of the driver. So, the resultant acceleration is 90° – 83° = 7° offset this direction towards the front of the car. Note that $\tan\theta$ is close to 10.

K4. Curveballs and Fastballs.

Baseball Courtesy Peter Miller, flickr
Attribution-NonCommercial-NoDerivs

Excellent benchmarks in the major leagues are given below for fastballs and curveballs, where rpm indicates rotations per minute.

Fastball: 2,500 rpm spin
Speed: 153.0 km/h (95.1 mph)

Curveball: 3,000 rpm spin
Speed: 129.0 km/h (80.2 mph)

Assuming a straight line path and no friction, how many rotations occur as the ball travels 18.44 m (60.50 ft) to the plate.

Fastballs. We will need the time of travel for the ball. We use $d = vt$, the only kinematic formula we will need. We do not need the rotational formulas! I want to work in meters per second, so first

$$v = 153.0 \ \frac{\text{km}}{\text{h}} = 153.0 \frac{1000 \text{ m}}{3600 \text{ s}} = 153.0 \frac{5}{18} \frac{\text{m}}{\text{s}} = 42.5000 \frac{\text{m}}{\text{s}} .$$

Then, $t_{fast} = \frac{d}{v} = \frac{18.44 \text{ m}}{42.5000 \frac{\text{m}}{\text{s}}} = 0.43388 \text{ s}$.

The number of rotations is

$$n_{fast} = 2500 \ \frac{\text{rotations}}{\text{minute}} \cdot 0.43388 \text{ s} .$$

Rotations do not have a dimension so we can write

$$n_{fast} = 2500 \ \frac{1}{\text{minute}} \cdot 0.43388 \text{ s} ,$$

i.e., 2500 per minute for 2500 rpm, where rotations are understood.

$$n_{fast} = 2500 \ \frac{1}{60 \text{ s}} \cdot 0.43388 \text{ s}$$

$$n_{fast} = 18.08$$

$$\boxed{n_{fast} = 18}$$

Curveballs.

$$v = 129.0 \ \frac{\text{km}}{\text{h}} = 129.0 \frac{1000 \text{ m}}{3600 \text{ s}} = 129.0 \frac{5}{18} \frac{\text{m}}{\text{s}} = 35.8333 \frac{\text{m}}{\text{s}}$$

$$t_{curve} = \frac{d}{v} = \frac{18.44 \text{ m}}{35.8333 \frac{\text{m}}{\text{s}}} = 0.51460 \text{ s}$$

The number of rotations is $n_{fast} = 3000 \ \frac{\text{rotations}}{\text{minute}} \cdot 0.51460 \text{ s}$.

$$n_{fast} = 3000\ \frac{1}{60\text{ s}} \cdot 0.51460\text{ s} = 25.73 \qquad \boxed{n_{curve} = 26}$$

K5. Playground Merry-Go-Round.

“Hold On” Courtesy Derek Bridges, flickr. License: Attribution 2.0 Generic.

Problem. A merry-go-round takes 2 seconds to go from rest to spinning at 30 rpm. Then, it takes a full minute to come to rest. (i) What is the average angular acceleration during the speed-up phase? (ii) What is the maximum angular velocity reached? (iii) Through what angle θ does the merry-go-round turn during this phase, approximating the acceleration as being constant? (iv) What is the average deceleration during the slow-down phase? (v) Through what angle θ does the merry-go-round turn during this latter phase, assuming constant deceleration?

Solution. (i) Speed-up phase. What is α ? The initial angular velocity is $\omega_0 = 0$ and the final angular velocity is $\omega = 30 \cdot 2\pi \frac{\text{rad}}{60\text{ s}} = \pi \frac{\text{rad}}{s}$.

$$\omega = \omega_0 + \alpha t \quad => \quad \omega = \alpha t \quad => \quad \alpha = \frac{\omega}{t} = \pi \frac{\text{rad}}{\text{s}} \frac{1}{2\text{ s}} = \frac{\pi}{2} \frac{\text{rad}}{\text{s}^2} \quad => \quad \boxed{\alpha = 1.57 \frac{\text{rad}}{\text{s}^2}}$$

(ii) Maximum angular velocity reached.

$$\omega = \omega_0 + \alpha t \quad => \quad \omega = \alpha t = \frac{\pi}{2} \frac{\text{rad}}{\text{s}^2} \cdot 2\text{ s} = \pi \frac{\text{rad}}{\text{s}} \quad => \quad \boxed{\omega = 3.14 \frac{\text{rad}}{\text{s}}}$$

(iii) Total angle for speed-up phase. Use $2\alpha(\theta-\theta_0)=\omega^2-\omega_0^2$ with $\theta_0=0$, $\omega_0=0$, and data from above answers for α and ω. Note that the answer below indicates 180°, a half rotation.

$$2\alpha\theta=\omega^2 \quad => \quad \theta=\frac{\omega^2}{2\alpha}=(\pi\frac{\text{rad}}{\text{s}})^2/(2\cdot\frac{\pi}{2}\frac{\text{rad}}{\text{s}^2})=\frac{\pi^2}{\pi}\text{rad}=\pi\ \text{rad} \quad => \quad \boxed{\theta=3.14\ \text{rad}}$$

(iv) Slow-down phase. What is α? The initial angular velocity is $\omega_0=\pi\frac{\text{rad}}{\text{s}}$ and the final angular velocity is $\omega=0$. The time to stop is $t=60\ \text{s}$.

$$\omega=\omega_0+\alpha t \quad => \quad 0=\pi\frac{\text{rad}}{\text{s}}+\alpha(60\ \text{s}) \quad => \quad \alpha=-\pi\frac{\text{rad}}{\text{s}}\frac{1}{60\ \text{s}}=\frac{\pi}{60}\frac{\text{rad}}{\text{s}^2} \quad => \quad \boxed{\alpha=0.052\frac{\text{rad}}{\text{s}^2}}$$

(v) Total angle for slow-down phase. Use $2\alpha(\theta-\theta_0)=\omega^2-\omega_0^2$ with $\theta_0=0$, $\omega=0$, and data from above answers (iv) and (iii) for α and ω_0. Note that the answer is 15 rotations! Each rotation is 2π and we have 30π below.

$$2\alpha\theta=-\omega_0^2 \quad => \quad \theta=-\frac{\omega_0^2}{2\alpha}=(\pi\frac{\text{rad}}{\text{s}})^2/(2\cdot\frac{\pi}{60}\frac{\text{rad}}{\text{s}^2})=\frac{\pi^2}{\pi/30}\text{rad}=30\pi\ \text{rad} \quad => \quad \boxed{\theta=94.2\ \text{rad}}$$

K6. Coefficient of Static Friction: Sushi on Vinyl.

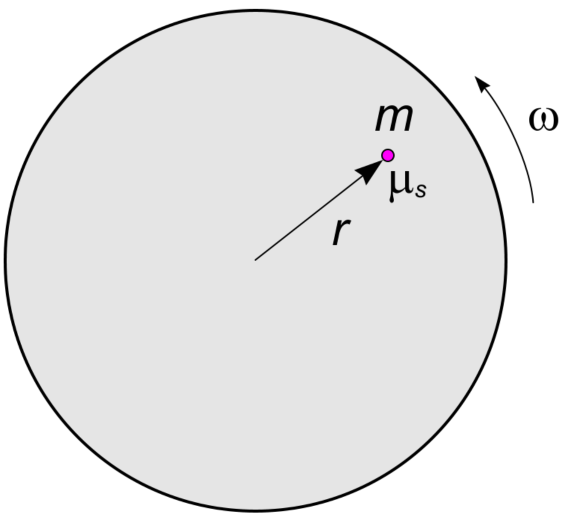
Sushi Turntable Courtesy Shunichi kouroki, flickr. [License: Attribution 2.0 Generic.]

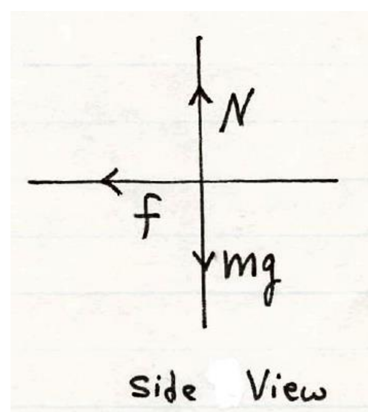
Problem. An experimenter finds on a variable turntable that at 80 rpm a specific 8-g piece of sushi at a distance 3 cm from the axis begins to slide off the spinning vinyl record. First convert your rpm to rad/s so that you have the angular velocity ω as one of your basic parameters.

(i) What is the coefficient of static friction for this sushi-vinyl surface in terms of the basic variables in the problem and numerically for the data given?

(ii) The experimenter checks the result by orienting the vinyl disk on an incline with the same piece of sushi on it. The angle that the disk makes with the horizontal is increased so that the sushi just begins to slide. What angle θ should the observer measure? Express the angle in the simplest way in terms of parameters in the problem. Then give the angle in degrees, based on the supplied data.

Solution. A sketch and force diagram are below. The force diagram is a side view looking along the edge of the disk, where the force f points towards the center.





(i) $\mu_s = ?$ The equations for motion are

$$f = m\frac{v^2}{r},$$

$$N - mg = 0.$$

With the friction equation, we have three equations.

$$f = m\frac{v^2}{r} \qquad N = mg \qquad f = \mu_s N$$

The last equation, with the second substituted in gives $f = \mu_s mg$. Then using the first equation

$$f = m\frac{v^2}{r},$$

becomes

$$\mu_s mg = m\frac{v^2}{r},$$

$$\mu_s g = \frac{v^2}{r},$$

$$\mu_s = \frac{v^2}{gr}.$$

But this equation is not good enough since v is not one of the starting variables.

Note that $v = \omega r$ and ω was given at the start as a variable. Then,

$$\mu_s = \frac{v^2}{gr} = \frac{(\omega r)^2}{gr} = \frac{\omega^2 r^2}{gr} = \frac{\omega^2 r}{g}$$

$$\boxed{\mu_s = \frac{\omega^2 r}{g}}$$

The mass of the piece of sushi does not matter. We have the coefficient of static friction for the sushi-vinyl surface.

For the numbers, we need to first convert 80 rpm to rad/s.

$$\omega = \frac{80 \cdot 2\pi}{60}\frac{\text{rad}}{\text{s}} = \frac{4}{3}2\pi\frac{\text{rad}}{\text{s}} = \frac{8}{3}\pi\frac{\text{rad}}{\text{s}}.$$

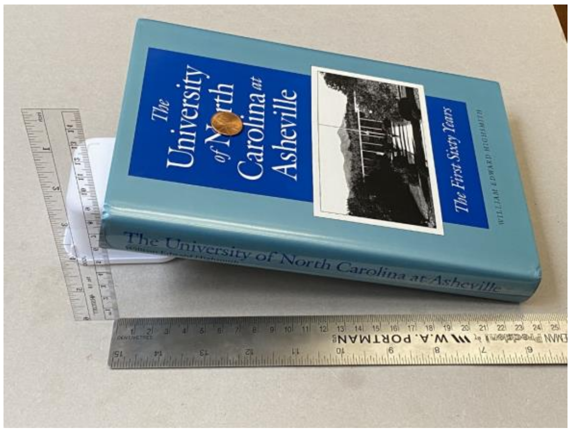
Remember that radians are technically dimensionless. Therefore we leave off the rad below.

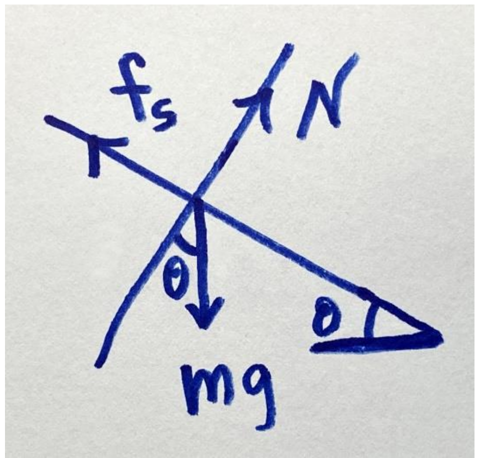
$$\mu_s = \frac{\omega^2 r}{g} = (\frac{8}{3}\pi\frac{1}{\text{s}})^2\frac{r}{g} = (\frac{8}{3}\pi\frac{1}{\text{s}})^2\frac{0.03\text{ m}}{9.8\text{ m/s}^2}$$

$$\mu_s = 70.184\frac{1}{\text{s}^2}\cdot\frac{3}{980}\text{s}^2 = 0.21485$$

$$\boxed{\mu_s = 0.21}$$

(ii) $\theta = ?$ We did this part of the problem before in an earlier chapter. But we need to reproduce it here as part of our new problem. It will be a good review.





The equations are

$$\sum F_{\text{down incline}} = mg\sin\theta - f_s = ma = 0$$

$$\sum F_{\text{normal to incline}} = N - mg\cos\theta = 0$$

Friction Equation: $f_s = \mu_s N$

The maxed out condition is met when the coin budges and at this threshold, i.e., right before it, there is still no motion ($a = 0$). The last two equations give

$$f_s = \mu_s N = \mu_s mg\cos\theta \,.$$

Using this equation in $mg\sin\theta - f_s = 0$, leads to $mg\sin\theta - \mu_s mg\cos\theta = 0$.

$$mg\sin\theta = \mu_s mg\cos\theta$$

The mg divides out.

$$\sin\theta = \mu_s\cos\theta$$

The coefficient of static friction is given by the tangent.

$$\boxed{\tan\theta = \mu_s}$$

Now since $\mu_s = \frac{\omega^2 r}{g}$ for the first part of this problem, we combine the results

$$\mu_s = \frac{\omega^2 r}{g} = \tan\theta$$

$$\boxed{\theta = \tan^{-1}(\frac{\omega^2 r}{g})}$$

Since we know $\mu_s = \frac{\omega^2 r}{g} = 0.21485$ from earlier, we can readily obtain the angle.

$$\theta = \tan^{-1}\mu_s$$

$$\theta = \tan^{-1} 0.21485$$

$$\theta = 12.126°$$

$$\boxed{\theta = 12°}$$

A nice problem with general formulas and specific numerical results.

The best of both worlds!