M0. Rotational Dynamics Strategy. First let's review the steps we take when solving problems in translational dynamics and be philosophical. In such cases, we usual have motion in one or two directions. Of course, we can have motion in three dimensions, but in introductory physics courses many of the problems involve two dimensions. Our approach has been to use four steps as follows.

(i) The Sketch. We first make a sketch of the problem.

(ii) The Force Diagrams. We then make a force diagram for each mass.

We place the masses at the origin in such diagrams and then draw in the forces. These diagrams with the mass freely placed at the origins are called free body diagrams. However, you can draw the forces on the actual sketched diagram of the first step. I will do this often for rotational dynamics since the positions of the forces are important for calculating the torques.

(iii) Equations of Motions. Then it is time for applying Newton's Second Law for each mass.

$$\sum F_x = ma_x$$

$$\sum F_y = ma_y$$

We choose our axes so that one of the accelerations in the above equations is zero.

Also in this step, there could be a needed auxiliary equation such as

$$f = \mu_k N \ .$$

(iv). Solving the Equations. Next comes the algebra. We solve the equations of step (iii), often for the acceleration and other parameters like tensions in ropes.

I like to encase the above within a general guiding principle for all problems in any field, where there is a question, an application of laws, a reflection on the answer, and communication. These broader steps are summarized in the mnemonic Inquiry-ARC pr I-ARC, where I is the inquiry or question, A is the application, R is the reflection, and C is communication.

In physics, the I is the given – you have a question and given parameters with symbols and often specific values. For the A, we have application of the laws. You can think of the four steps given above (i to iv) as the details for our application. Then when we get the answer we are not finished. We need to R = reflect on our answer to verify and certify that it is correct. Often, I derive a general formula for the situation, even though numbers may be supplied in the given. I will typically insert the values at the end. But before doing so, I check extreme cases with the formulas. Perhaps I take one mass to be infinitely large and see what is predicted. These extreme values will be simpler versions of the problem and one where you know the answer. You check your answer in this way.

When I was a graduate student at the University of Maryland, my mentor Prof. David Falk would say he has to check his answers. He would often do the problem another way. As a teaching assistant in graduate school I would hope the problem was an odd-numbered problem since the book would typically have answers to the odd problems at the back of the book. That approach is a weak one of course. So, two ways to reflect on your analysis and answer are these two below.

I. Examine Extreme Cases.

Let a parameter take on extreme values and see if your formulas make sense.

II. Solve the Problem Another Way.

Richard Feynman once said something to the effect that a good physicist can see a problem from about 5 or 7 approaches.

Of course, there is nothing wrong with checking the answer with one in the book or another book, or googling. Scientists often work in teams. But the two gold standards for reflecting on your work are the extreme cases and the alternative solutions.

Finally, for C, the communication, I make sure my dimensions are correct. If I am solving for a force, the units need to be newtons or a unit for force. I will usually perform this step first as part of my reflection, but I like to list it under communication so I can have two golden rules for communication.

1. Are the Dimensions Correct?

2. Do I Have the Correct Amount of Significant Figures?

Significant figures are important. If values are supplied in the problem, I need to examine how many significant figures are given for the parameters. The numerical answer must be consistent with the number of significant figures. The Chemistry Rule is to strictly use the amount of significant figures for the weakest link, i.e., the amount of significant figures for the most uncertain parameter. The Engineering Practice is to give three significant matters even if one parameter is given with two significant figures. Ask your teacher what you should do. When I taught Physics, I would be okay with students using either the Chemistry Rule or the Engineering practice. Lots of books these days are giving the decimal point and zero such as 20.0 kg or 50.0 m/s to clearly communicate significant figures at the start.

For the rotational problems, the steps are all the same with the addition of the torque equation and a new auxiliary equation that relates the translational acceleration to the rotational acceleration $a = \alpha R$.

What is this additional torque equation we are going to add? Let's arrive at an approach in general. We start with the following equations from Chapter K Rotation.

$$\vec{\tau} = \frac{d\vec{l}}{dt} \qquad \vec{\tau} = \vec{r} \times \vec{F} \qquad \vec{l} = \vec{r} \times \vec{p}$$

We will study in this chapter solid objects made of mass elements all rotating around a common axis. Therefore, we can drop the vector sign but we need to consider the object as having mass elements since bits of mass at different distances from the axis of rotation will give different rotational inertias.

$$\tau = \frac{dl}{dt} \qquad l = \sum_i r_i p_i = \sum_i r_i m_i v_i$$

Since all the mass points rotate about a common axis with angular velocity and angular acceleration satisfying the equations $v_i = \omega r_i$ and $a_i = \alpha r_i$, we can write

$$l = \sum_i r_i m_i v_i = \sum_i r_i m_i (\omega r_i) = \sum_i r_i^2 m_i \omega = (\sum_i m_i r_i^2)\,\omega = I\omega$$

Then, when we apply $\tau = \frac{dl}{dt}$ for the fixed rotational inertia, we find

$$\tau = \frac{dl}{dt} \quad => \quad \tau = \frac{d}{dt}(I\omega) \quad => \quad \tau = I\frac{d\omega}{dt} \quad => \quad \tau = I\alpha$$

Think of the last equation as being analogous to $F = ma$. The three basic equations for rotational dynamics are then simply

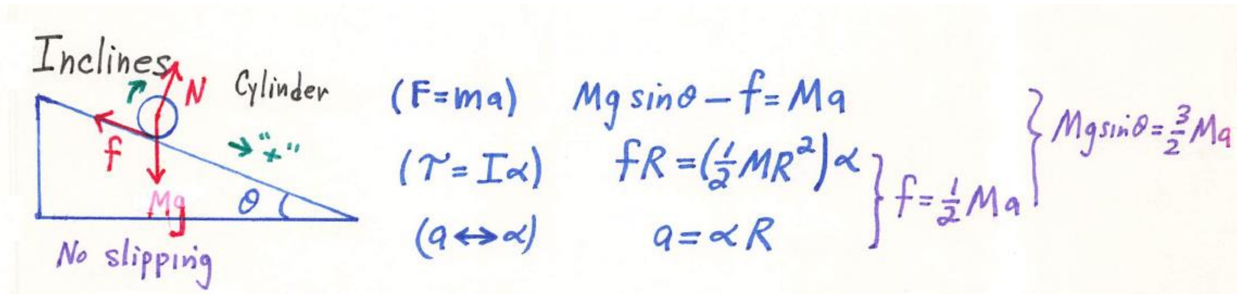
$$\sum F = ma\,,$$

$$\sum \tau = I\alpha\,,$$

$$a = \alpha R\,.$$

You will also need to use the formula for the moment of inertia specific to your problem. The best way to see how all this works is to see an example. The next section will serve our purpose here. Actually, the rest of the chapter consists of examples. There are 10 in all and they cover the repertoire of traditional problems studies and assigned in rotational dynamics.

M1. Rolling Down an Incline. In this section we consider objects that roll down an incline without slipping. First consider a rolling cylinder. The acceleration will be found to be $a = \frac{2}{3} g \sin\theta$. See the figure below for the steps. The cylinder mass is M and the moment of inertia is $I = \frac{1}{2} MR^2$.



The forces have been added to the sketch. These forces are the force due to gravity, the normal force, and the friction which allows the cylinder to roll rather than slip.

Then we sum the forces down the incline.

$$\sum F = Mg\sin\theta - f = Ma$$

We can go right to the torque equation.

$$\sum \tau = f R = I\alpha = (\frac{1}{2} MR^2)\alpha$$

The auxiliary equation is the connecting equation

$$a = \alpha R .$$

I like to work from the third equation up like we did for friction problems when the auxiliary equation related the friction to the normal force. Here we do not need such a friction equation. Watch how friction will drop out. First substitute $\alpha = \frac{a}{R}$ into $f R = \frac{1}{2} MR^2 \alpha$:

$$f R = \frac{1}{2} MR^2 \frac{a}{R} .$$

Watch how the radius drops out.

$$f = \frac{1}{2} Ma$$

Now substitute this equation in the first one $Mg\sin\theta - f = Ma$ to obtain

$$Mg\sin\theta - \frac{1}{2}Ma = Ma \; .$$

The masses drop out.

$$g\sin\theta - \frac{1}{2}a = a$$

Solve for the acceleration.

$$g\sin\theta = a + \frac{1}{2}a$$

$$g\sin\theta = \frac{3}{2}a$$

$$a = \frac{2}{3}g\sin\theta$$

Another interesting question is what is the speed at the bottom of the ramp is the cylinder starts from rest and the height of the ramp is h. We use conservation of energy, remembering that kinetic energy has two parts: translational and rotational. Let position 1 be at the top of the incline and position 2 be at the bottom. Conservation of energy leads to

$$K_1 + U_1 = K_2 + U_2 \; ,$$

$0 + Mgh = K_2 + U_2$, where

$K_2 = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$ and $U_2 = 0$.

These relations lead to

$$0 + Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 + 0 \; .$$

The connecting equation $v = \omega R$ is important for the next step, along with $I = \frac{1}{2}MR^2$.

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I(\frac{v}{R})^2$$

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}(\frac{1}{2}MR^2)(\frac{v}{R})^2$$

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{4}Mv^2$$

$$Mgh = \frac{3}{4}Mv^2$$

$$gh = \frac{3}{4}v^2$$

$$v = \sqrt{\frac{4}{3}gh}$$

We would like to repeat this calculation for the hoop, solid sphere, and spherical shell. But rather than perform the calculation three more times, we note that the moments of inertia have the form

$$I = \varepsilon MR^2 ,$$

where ε is a value such as 1 (hoop), $\frac{1}{2}$ (cylinder), $\frac{2}{5}$ (sphere), and $\frac{2}{3}$ (spherical shell).

We do the rolling problem in general now with $I = \varepsilon MR^2$. The three basic equations are below.

$$\sum F = Mg\sin\theta - f = Ma$$

$$\sum \tau = fR = I\alpha = (\varepsilon MR^2)\alpha$$

$$a = \alpha R$$

We solve these equations like before with the ε in there.

Substitute $\alpha = \frac{a}{R}$ into $fR = (\varepsilon MR^2)\alpha$, to find

$$fR = (\varepsilon MR^2)\frac{a}{R},$$

$$f = \varepsilon Ma .$$

Then, this equation is used in $Mg\sin\theta - f = Ma$.

The result is

$$Mg\sin\theta - \varepsilon Ma = Ma\,.$$

The masses drop out,

$$g\sin\theta - \varepsilon a = a\,,$$

and we readily solve for the acceleration,

$$g\sin\theta = a + \varepsilon a$$

$$g\sin\theta = a(1+\varepsilon)$$

$$a = \frac{g\sin\theta}{1+\varepsilon}$$

For the velocity at the bottom

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I(\frac{v}{R})^2$$

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}(\varepsilon MR^2)(\frac{v}{R})^2$$

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}\varepsilon Mv^2$$

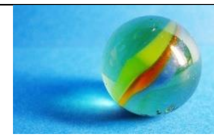
$$gh = \frac{1}{2}v^2 + \frac{1}{2}\varepsilon v^2$$

$$2gh = v^2 + \varepsilon v^2$$

$$2gh = v^2(1+\varepsilon)$$

$$v = \sqrt{\frac{2gh}{1+\varepsilon}}$$

Now we can make a table with all the results.

				
Object	Ring	Battery	Marble	Ping-Pong
Physics	Hoop	Cylinder	Sphere	Shell
Rotational Inertia	MR^2	$\frac{1}{2}MR^2$	$\frac{2}{5}MR^2$	$\frac{2}{3}MR^2$
ε	1	$1/2$	$2/5$	$2/3$
$a = \frac{g\sin\theta}{1+\varepsilon}$	$\frac{1}{2}g\sin\theta$	$\frac{2}{3}g\sin\theta$	$\frac{5}{7}g\sin\theta$	$\frac{3}{5}g\sin\theta$
$v = \sqrt{\frac{2gh}{1+\varepsilon}}$	$\sqrt{gh}$	$\sqrt{\frac{4}{3}gh}$	$\sqrt{\frac{10}{7}gh}$	$\sqrt{\frac{5}{4}gh}$
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Which object wins the race down the incline? The winner is the solid sphere. The order of the competitors in the are:

Sphere (1st place), Cylinder (2nd place), Spherical Shell (3rd place), and Hoop (4th place).

When the mass "hugs" the axis of rotation, you have a faster rolling object!

M2. A Car and the Incline. When I taught this chapter in class I would start the class asking the students which object would win a race down an incline. I brought a wooden board to class with a marble, cylinder (tinker toy cylinder, a ping-pong ball, and a hot-wheels car. We took votes and then had races with two objects at a time. The objects then eliminated each other. I saved the car for last since I knew it would win.

HotWheels – Ferrari 458 Italia

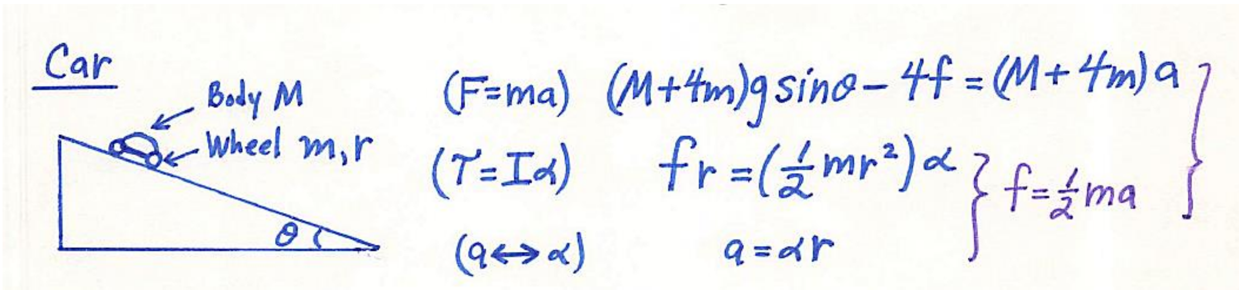
Courtesy Leap Kye, flickr. License: Attribution-NoDerivs 2.0 Generic.

The secret as to why the car wins is that only the wheels down. The body of the car adds mass to the weight due to the pull of gravity. We will work out a general formula where each wheel has mass m and the body of the car (minus the wheels) has mass M. The total mass of the car is then

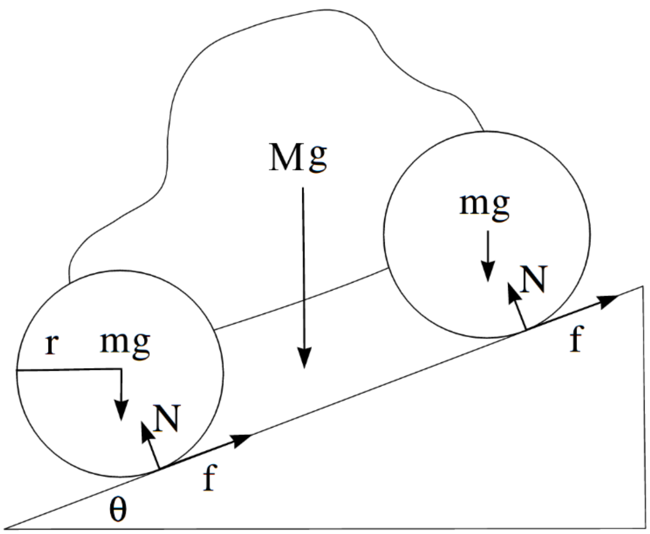
$$M_{car} = M + 4m \, .$$

The following figure illustrates the calculation.

The figure comes from my old notes c. 1980.



Note how working from the bottom up is a workable approach. I will start from scratch below and show every step. The figure I have below is flipped. It doesn't matter which way the ramp faces.



First, $F = ma$ leads to

$$(M+4m)g\sin\theta - 4f = (M+4m)a,$$

where there are four wheels, each with mass m.

Second, the torque equation $\tau = I\alpha$, gives

$$fr = I\alpha = \frac{1}{2}mr^2\alpha$$

for each wheel. The connecting equation is $a = \alpha r$.

The three equations are

$$(M+4m)g\sin\theta - 4f = (M+4m)a,$$

$$fr = \frac{1}{2}mr^2\alpha,$$

$$\alpha = \frac{a}{r}.$$

Working from the bottom up, substitute $\alpha = \frac{a}{r}$ into $fr = \frac{1}{2}mr^2\alpha$ to obtain

$$fr = \frac{1}{2} mr^2 \frac{a}{r}\,.$$

The r parameters will cancel.

$$f = \frac{1}{2} ma$$

Then, we substitute this equation in $(M + 4m) g \sin\theta - 4f = (M + 4m)a$.

$$(M + 4m) g \sin\theta - 4(\frac{1}{2} ma) = (M + 4m)a$$

Solve for the acceleration.

$$(M + 4m) g \sin\theta = (M + 4m)a + 4(\frac{1}{2} ma)$$

$$(M + 4m) g \sin\theta = (M + 4m)a + 2ma$$

$$(M + 4m) g \sin\theta = (M + 6m)a$$

$$a = \left[\frac{M + 4m}{M + 6m}\right] g \sin\theta$$

Dimensions look good as the mass ratio is dimensionless and we are left with acceleration g.

Check when the mass of the body of the car vanishes. Do we get the cylinder result?

$$\lim_{M \to 0} a = \lim_{M \to 0} \left[\frac{M + 4m}{M + 6m}\right] g \sin\theta = \left[\frac{0 + 4m}{0 + 6m}\right] g \sin\theta = \frac{4m}{6m} g \sin\theta = \frac{2}{3} g \sin\theta$$

It checks out.

What about when the wheel masses go to zero.

$$\lim_{m \to 0} a = \lim_{m \to 0} \left[\frac{M + 4m}{M + 6m}\right] g \sin\theta = \left[\frac{M + 0}{M + 0}\right] g \sin\theta = \frac{M}{M} g \sin\theta = g \sin\theta$$

Wow! HotWheels cars with their small wheels will be the winner!

The HotWheels car will beat our other objects.

In fact, it will beat all objects where the entire object rolls!

This conclusion follows by comparing with $a = \dfrac{g\sin\theta}{1+\varepsilon}$.

For rolling masses ($I = \varepsilon MR^2$) the parameter $\varepsilon > 0$.

Therefore,

$$a = \frac{g\sin\theta}{1+\varepsilon} < g\sin\theta .$$

The car with tiny wheels approaches the acceleration of a block sliding down an incline with no friction. I remember seeing a physics problem book from the 1970s where there they had problems where they made the rolling cars with tiny wheels. They did these diagrams to achieve the equivalent of a mass sliding with no friction. The car with tiny wheels rolling on a surface with friction to get those wheels turning is more realistic. The book was *General Physics Workbook: Physics Problems and How to Solve Them* by Foster Strong at Caltech (W. H. Freeman and Company, San Francisco, 1972).

Standing in Race	Object	Acceleration a	$\dfrac{a}{g\sin\theta}$
1st Place	Hot Wheels Car	$\approx g\sin\theta$	≈ 1.00
2nd Place	Sphere	$\frac{5}{7}g\sin\theta$	0.70
3rd Place	Cylinder	$\frac{2}{3}g\sin\theta$	0.67
4th Place	Shell	$\frac{3}{5}g\sin\theta$	0.60
5th Place	Hoop	$\frac{1}{2}g\sin\theta$	0.50

I did not have a hoop in class. I would be hard to roll a ring. But you could roll a hollow cylinder. That would be the same as a hoop. I didn't have one at the time.

One last thing is the find the speed of the car at the bottom of a ramp. Since the car would hit the table, we would use a height h that doesn't exactly reach the table.

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\left(\frac{v}{R}\right)^2$$

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}(\varepsilon MR^2)(\frac{v}{R})^2$$

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}\varepsilon Mv^2$$

$$gh = \frac{1}{2}v^2 + \frac{1}{2}\varepsilon v^2$$

$$2gh = v^2 + \varepsilon v^2$$

$$2gh = v^2(1+\varepsilon)$$

$$v = \sqrt{\frac{2gh}{1+\varepsilon}}$$

We start with conservation of energy.

$$K_1 + U_1 = K_2 + U_2 \, .$$

where

$$K_1 = 0 \, ,$$

$$U_1 = (M + 4m)gh \, ,$$

$$K_2 = \frac{1}{2}M_{car}v^2 + 4\frac{1}{2}I_{wheel}v^2 \, ,$$

$$U_2 = 0 \, .$$

Conservation of energy is then

$$(M + 4m)gh = \frac{1}{2}(M + 4m)v^2 + 4(\frac{1}{2}I\omega^2)$$

$$(M + 4m)gh = \frac{1}{2}(M + 4m)v^2 + 4\left[\frac{1}{2}(\frac{1}{2}mr^2)\omega^2\right]$$

$$(M + 4m)gh = \frac{1}{2}(M + 4m)v^2 + mr^2\omega^2$$

$$(M+4m)gh = \frac{1}{2}(M+4m)v^2 + mv^2$$

$$(M+4m)gh = \frac{1}{2}(M+6m)v^2$$

$$2(M+4m)gh = (M+6m)v^2$$

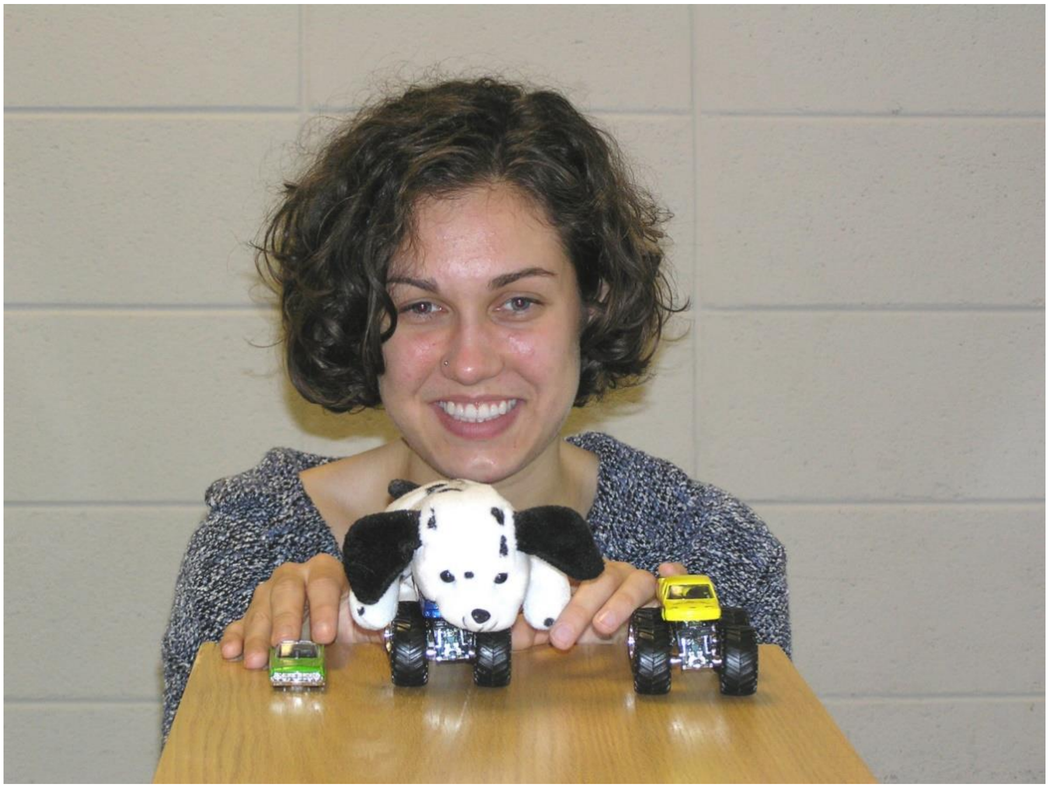
$$v = \sqrt{\frac{2(M+4m)gh}{(M+6m)}}$$

$$v = \sqrt{\frac{M+4m}{M+6m}}\sqrt{2gh}$$

When you can neglect the wheels, you get $v = \sqrt{2gh}$ the result for falling through height h.

Do you get the cylinder result when $M = 0$?

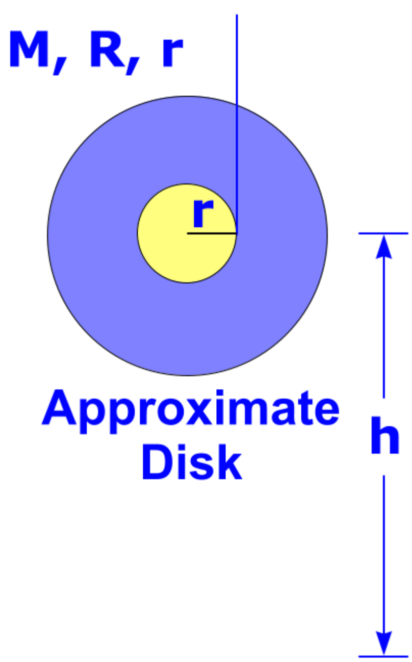
My Hot Wheels car with tiny wheels is the left one in the figure below.



Hot Wheels Car with Tiny Wheels at the Left with a Daughter. Photo by doctorphys.

M3. Yo-yo. The yo-yo, also spelled yoyo, is a nice example of rotational dynamics.

Courtesy Enrique Calabuig, flickr. License: Attribution-ShareAlike 2.0 Generic



We model the yo-yo as a cylinder where we neglect slight deviations in shape and neglect the narrow gap to make room for the string.

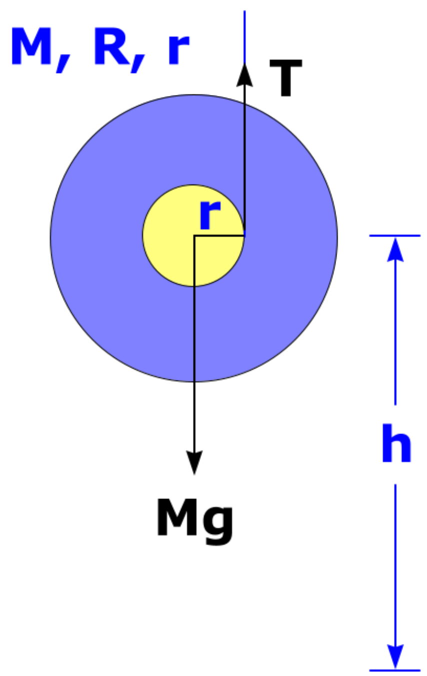
We want to find the acceleration for the yo-yo and the velocity if it falls a distance h from rest.

The yo-yo has two radii for us to consider. The inner radius r is where the string acts on the yo-yo. The outer radius R is the larger radius that takes us to the edge of the yo-yo. The mass of the yo-yo is M .

The torque on the yo-yo is equal to the tension in the string multiplied by the radius r. Note that the tension force is perpendicular to the radius. Therefore,

$$\vec{\tau} = \vec{r} \times \vec{F} = rF \sin 90° n = rF n \, ,$$

where F is the force on the string pulling up and the unit vector n points out of the page or computer monitor. The motion is counterclockwise. You can curve your right-hand fingers along the circle and your thumb will point out of the page, i.e., in the direction of n.



The forces have been added in the diagram at the left. We proceed to the three equations: (i) $F = ma$, (ii) $\tau = I\alpha$, and (iii) $a = \alpha r$, where for the yo-yo $I = \frac{1}{2}MR^2$.

$$\text{(i) } Mg - T = Ma$$

$$\text{(ii) } rT = \frac{1}{2}MR^2\alpha$$

$$\text{(iii) } \alpha = \frac{a}{r}$$

Why is the small radius the one to use here?

Start at the bottom as usual. Substitute (iii) in (ii).

$$rT = \frac{1}{2}MR^2\frac{a}{r}$$

Usually R drops out, but will not here since there are two radii.

$$T = \frac{1}{2}M\frac{R^2}{r^2}a$$

Now we use this result in the (i). Then,

$$Mg - T = Ma \text{ becomes } Mg - \frac{1}{2}M\frac{R^2}{r^2}a = Ma.$$

Solve for the acceleration.

$$Mg = Ma + \frac{1}{2}M\frac{R^2}{r^2}a$$

The mass drops out.

$$g = a + \frac{1}{2}\frac{R^2}{r^2}a$$

$$g = (1 + \frac{1}{2}\frac{R^2}{r^2})a$$

$$a = \frac{g}{1 + \frac{1}{2}\frac{R^2}{r^2}}$$

Let’s do a quick dimensions check. The dimensions are correct since $\frac{R}{r}$ is dimensionless and the dimensions of acceleration a match with the dimensions of g. My preference is to clean the formula up a little by multiplying to and bottom by $2r^2$.

$$a = \frac{2r^2}{2r^2 + R^2} g$$

Is this answer reasonable? Let’s look into two extreme cases.

Case 1. The inner radius $r = 0$. In this case there is no longer any torque and the acceleration is

$$\lim_{r \to 0} a = \lim_{r \to 0} \frac{2r^2}{2r^2 + R^2} g = \frac{0}{0 + R^2} g = 0 .$$

The string connects at the center of the yo-yo and there is no motion.

Case 2. The inner radius $r = R$. In this extreme we have the unwrapping disk or cylinder.

$$\lim_{r \to R} a = \lim_{r \to R} \frac{2r^2}{2r^2 + R^2} g = \frac{2R^2}{2R^2 + R^2} g = \frac{2R^2}{3R^2} g = \frac{2}{3} g$$

Remember our rolling cylinder going down an include and the formula $a = \frac{2}{3} g \sin\theta$? The unwrapping disk is like taking θ to approach 90°. Then,

$$\lim_{\theta \to 90^\circ} \frac{2}{3} g \sin\theta = \frac{2}{3} g \sin 90^\circ = \frac{2}{3} g .$$

It checks out!

For the velocity after the yo-yo unwinds from rest through a distance height of h we use conservation of energy.

$$K_1 + U_1 = K_2 + U_2$$

$$0 + Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 + 0$$

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}(\frac{1}{2}MR^2)\omega^2$$

But we have to be careful with the angular velocity ω since $v = \omega r$, where the inner radius is used.

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}(\frac{1}{2}MR^2)(\frac{v}{r})^2$$

The mass still drops out by we are left with radii in the formula.

$$gh = \frac{1}{2}v^2 + \frac{1}{2}(\frac{1}{2}R^2)(\frac{v}{r})^2$$

$$gh = \frac{1}{2}v^2 + \frac{1}{2}(\frac{1}{2}\frac{R^2}{r^2})v^2$$

$$2gh = v^2 + \frac{1}{2}\frac{R^2}{r^2}v^2$$

$$2gh = (1 + \frac{1}{2}\frac{R^2}{r^2})v^2$$

$$v = \sqrt{\frac{2gh}{1 + \frac{1}{2}\frac{R^2}{r^2}}}$$

$$v = \sqrt{\frac{1}{1 + \frac{1}{2}\frac{R^2}{r^2}}}\sqrt{2gh}$$

$$v = \sqrt{\frac{2r^2}{2r^2 + R^2}}\sqrt{2gh}$$

Is this answer reasonable? The dimensions check out. Why?

Another check is to do the problem another way. We know the acceleration formula

$$a = \frac{2r^2}{2r^2 + R^2} g \ .$$

We can get the final velocity from our kinematic formula

$$2ad = v^2 - v_0^2 ,$$

where $d = h$ and $v_0 = 0$.

Then,

$$2\left[\frac{2r^2}{2r^2 + R^2} g \right] h = v^2$$

$$v^2 = \left[\frac{2r^2}{2r^2 + R^2} \right] 2gh$$

$$v = \sqrt{\frac{2r^2}{2r^2 + R^2}} \sqrt{2gh}$$

It checks out!

What about a final check with our cylinder rolling down the incline?

From our incline section, the cylinder gives $v = \sqrt{\frac{4}{3} gh}$ with no dependence on the incline angle.

To compare formulas, we take $r = R$.

$$\lim_{r \to R} v = \lim_{r \to R} \sqrt{\frac{2r^2}{2r^2 + R^2}} \sqrt{2gh} = \sqrt{\frac{2R^2}{2R^2 + R^2}} \sqrt{2gh} = \sqrt{\frac{2R^2}{3R^2}} \sqrt{2gh}$$

$$\lim_{r \to R} v = \sqrt{\frac{2R^2}{3R^2}} \sqrt{2gh} = \sqrt{\frac{2}{3}} \sqrt{2gh} = \sqrt{\frac{4}{3} gh}$$

And again, the formulas check out.

We end this section with some numbers. The Duncan Yo-Yo has a mass of 67.9 grams and radius $R = 2.8 \text{ mm}$. An actual yo-yo does not have a constant inner radius r since as the string unwinds, the inner radius changes. We will make an approximation by taking $r = \frac{R}{4}$. The acceleration actually only depends on the ratio R/r if we consult the acceleration formula in the form

$$a = \frac{g}{1 + \frac{1}{2}\frac{R^2}{r^2}} .$$

Using $r = \frac{R}{4}$, i.e., $\frac{R}{r} = 4$,

$$a = \frac{g}{1 + \frac{1}{2}\frac{R^2}{r^2}} = \frac{g}{1 + \frac{1}{2}(4)^2} = \frac{g}{1 + \frac{16}{2}} = \frac{g}{1+8} = \frac{1}{9}g$$

$$a = \frac{1}{9}g = \frac{1}{9} \cdot 9.8 \ \frac{\text{m}}{\text{s}^2} = 1.09 = 1.1 \ \frac{\text{m}}{\text{s}^2}$$

I prefer the following form for the answer.

$$a = \frac{1}{9}g$$

The speed after falling one meter is

$$v = \sqrt{\frac{1}{1 + \frac{1}{2}\frac{R^2}{r^2}}}\sqrt{2gh} .$$

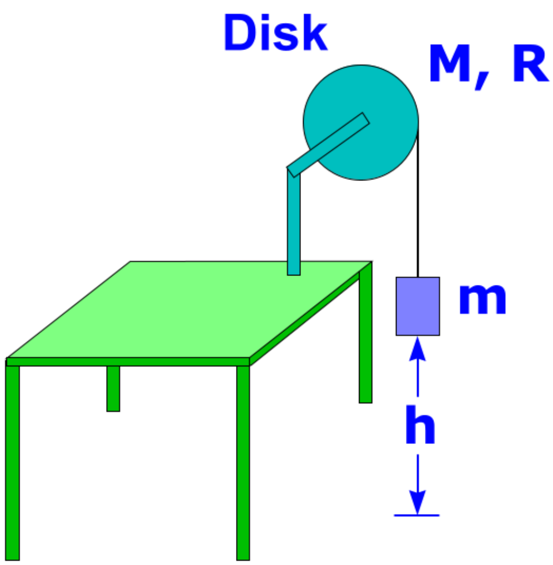
$$v = \sqrt{\frac{1}{1 + \frac{1}{2}4^2}}\sqrt{2g \cdot 1} = \sqrt{\frac{1}{1 + \frac{1}{2} \cdot 16}}\sqrt{2g} = \sqrt{\frac{1}{1+8}}\sqrt{2g} = \sqrt{\frac{2}{9}g} = \sqrt{\frac{2}{9} \cdot 9.8} = 1.48 = 1.5 \frac{\text{m}}{\text{s}}$$

How long does it take to get down there?

We can use $v = at$.

$$t = \frac{v}{a} = \frac{1.48}{1.09} = 1.36 = 1.4 \text{ s}$$

M4. Hanging Mass and Rotating Pulley.



A mass m is hanging and rotating a disk pulley of mass M and radius R as shown in the figure.

(a) What is the acceleration of the mass and tension in the rope?

(b) What is the velocity if the mass falls from rest through a distance of height h.

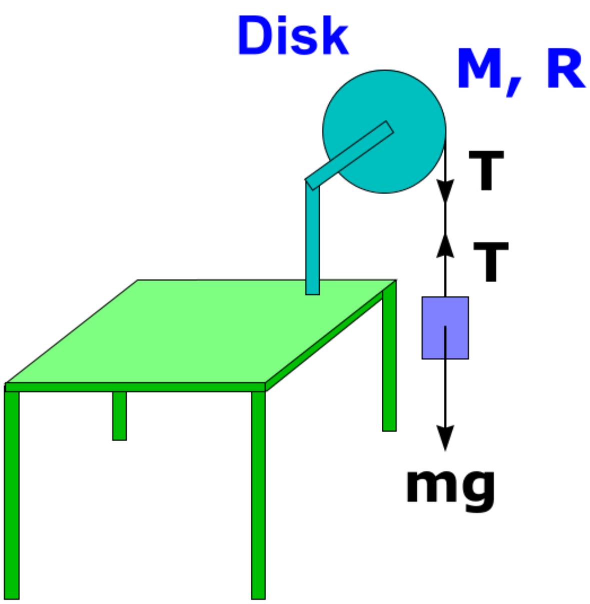
(c) What is the angular velocity ω at this point?

(d) What is the time to fall the distance h?

(e) Find the answers to the above parts for the values

$$M = 10 \text{ kg}, \ R = 30 \text{ cm}, \ m = 2 \text{ kg}, \ h = 50 \text{ cm}.$$

We start the analysis by labeling the forces.



(a) The acceleration. We proceed to the three equations: (i) $F = ma$, (ii) $\tau = I\alpha$, and (iii) $a = \alpha R$, where for the yo-yo $I = \frac{1}{2}MR^2$.

(i) $mg - T = ma$

(ii) $RT = \frac{1}{2}MR^2\alpha$

(iii) $\alpha = \frac{a}{R}$

Insert (iii) in (ii).

$$RT = \frac{1}{2}MR^2\frac{a}{R} \quad \Rightarrow \quad T = \frac{1}{2}Ma$$

Next substitute the tension $T = \frac{1}{2}Ma$ in (i).

$$mg - \frac{1}{2}Ma = ma \quad => \quad mg = ma + \frac{1}{2}Ma \quad => \quad mg = (m + \frac{1}{2}M)a$$

Solve for the acceleration.

$$mg = (m + \frac{1}{2}M)a \quad => \quad a = \frac{m}{m + \frac{1}{2}M} g \quad => \quad a = \frac{2m}{2m + M} g$$

$$a = \frac{2m}{2m + M} g$$

$$T = \frac{1}{2}Ma$$

$$T = \frac{mM}{2m + M} g$$

Are these answers reasonable?

The dimensions are correct since the mass dimensions cancel in the acceleration equation.

For the tension equation there is an extra mass factor giving a force.

We check some cases.

Case 1. Super Large Mass M .

$$\lim_{M \to \infty} a = \lim_{M \to \infty} \frac{2m}{2m + M} g = 0$$

The hanging mass just hangs there. The weight should be mg. Let's check.

$$\lim_{M \to \infty} T = \lim_{M \to \infty} \frac{mM}{2m + M} g = \lim_{M \to \infty} \frac{m}{\frac{2m}{M} + 1} g = \frac{m}{0 + 1} g = mg$$

Case 2. Super Small Mass M . The hanging mass will be in free fall with no string tension.

$$\lim_{M \to 0} a = \lim_{M \to 0} \frac{2m}{2m + M} g = \frac{2m}{2m + 0} g = g$$

$$\lim_{M\to 0} T = \lim_{M\to 0} \frac{mM}{2m+M} g = 0$$

Case 3. Super Small Mass m . Let the hanging mass be a feather. There is no acceleration and the mass hangs there with negligible tension in the rope:

$$\lim_{m\to 0} a = \lim_{m\to 0} \frac{2m}{2m+M} g = 0 \quad \text{and} \quad \lim_{m\to 0} T = \lim_{m\to 0} \frac{mM}{2m+M} g = 0 .$$

Case 4. Super Large Mass m . The hanging mass is in free fall causing the pulley to turn.

$$\lim_{m\to\infty} a = \lim_{m\to\infty} \frac{2m}{2m+M} g = \lim_{m\to\infty} \frac{2}{2+\frac{M}{m}} g = g \quad \text{and} \quad \lim_{m\to\infty} T = \lim_{m\to\infty} \frac{mM}{2m+M} g = \lim_{m\to\infty} \frac{M}{2+\frac{M}{m}} g = \frac{1}{2} Mg .$$

But since we are in free fall shouldn't the tension be zero?

We need to be careful with limits. The best way to see through this situation is to use values that are not infinity. Whenever in doubt with limits, use this trick. Let $m = 50M$ rather than infinity.

$$a = \frac{2m}{2m+M} g = \frac{2(50M)}{2(50M)+M} g = \frac{100M}{101M} g = \frac{100}{101} g \approx g$$

$$T = \frac{1}{2} Ma = \frac{1}{2} M \frac{100}{101} g = \frac{50}{101} Mg \approx \frac{1}{2} Mg$$

Now we check if $mg - T = ma$?

We now substitute $m = 50M$, $a = \frac{100}{101} g$, and $T = \frac{50}{101} Mg$.

$$mg - T = ma \quad \Rightarrow \quad 50Mg - \frac{50}{101} Mg = (50M) \frac{100}{101} g \,?$$

$$50 - \frac{50}{101} = (50) \frac{100}{101} ? \quad \Rightarrow \quad 50 = (50) \frac{100}{101} + \frac{50}{101} ? \quad \Rightarrow \quad 50 = (50) \left[\frac{100}{101} + \frac{1}{101} \right] ?$$

$$50 = (50) \left[\frac{101}{101} \right] ? \quad \Rightarrow \quad 50 = 50 \,?$$

The answer is yes and we are good!

There is no contradiction.

(b) The velocity. Conservation of energy:

$$K_1 + U_1 = K_2 + U_2$$

$$0 + mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + 0$$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}(\frac{1}{2}MR^2)\omega^2$$

The angular velocity ω is related to the linear velocity v by $v = \omega R$.

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}(\frac{1}{2}MR^2)(\frac{v}{R})^2$$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}(\frac{1}{2}M)v^2$$

$$4mgh = 2mv^2 + Mv^2$$

$$4mgh = (2m + M)v^2$$

$$v^2 = \frac{4mgh}{2m + M}$$

$$v = \sqrt{\frac{4mgh}{2m + M}}$$

(c) The angular velocity.

$$\omega = \frac{v}{R} \quad => \quad v = \frac{1}{R}\sqrt{\frac{4mgh}{2m + M}}$$

(d) The time.

$$t = \frac{h}{v} = \frac{h}{\sqrt{\frac{4mgh}{2m + M}}} = \sqrt{\frac{(2m + M)h}{4mg}}$$

(e) Numbers.

$$M = 10 \text{ kg},\ R = 30 \text{ cm},\ m = 2 \text{ kg},\ h = 50 \text{ cm}$$

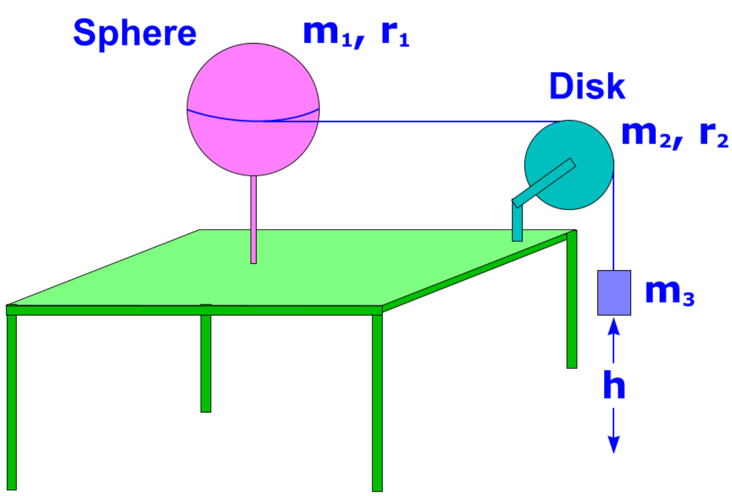
$$a = \frac{2m}{2m+M} g = \frac{2 \cdot 2}{2 \cdot 2 + 10} \cdot 9.8 = \frac{4}{14} 9.8 = 2.8 \ \frac{\text{m}}{\text{s}^2}$$

$$v = \sqrt{\frac{4mgh}{2m+M}} = \sqrt{\frac{4 \cdot 2 \cdot 9.8 \cdot 0.5}{2 \cdot 2 + 10}} = \sqrt{\frac{39.2}{14}} = 1.67 \ \frac{\text{m}}{\text{s}} = 1.7 \ \frac{\text{m}}{\text{s}}$$

$$\omega = \frac{v}{R} = \frac{1.67}{0.30} = 5.57 = 5.6 \frac{\text{rad}}{\text{s}}$$

$$t = \frac{h}{v} = \frac{0.3}{1.67} = 0.18 \text{ s}$$

M5. Dropping Mass with Rotating Pulley and Sphere.



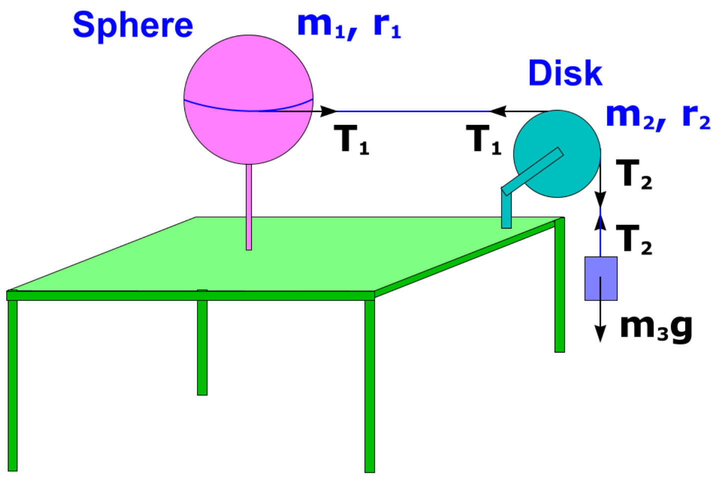
This problem is similar to the last one. We just added a rotating sphere to the system.

The questions are"

(a) What is the acceleration?

(b) What are the tensions in the cables?

(c) Find the speed when the hanging mass travels through a distance h starting from rest.



The equations are below.

$$\tau_1 = I_1\alpha_1 \quad => \quad T_1 r_1 = \frac{2}{5} m_1 r_1^2 \frac{a}{r_1}$$

$$\tau_2 = I_2\alpha_2 \quad => \quad (T_2 - T_1) r_2 = \frac{1}{2} m_2 r_2^2 \frac{a}{r_2}$$

$$F_3 = m_3 a \quad => \quad m_3 g - T_2 = m_3 a$$

Simplifying slightly brings us to

$$T_1 = \frac{2}{5} m_1 a$$

$$T_2 - T_1 = \frac{1}{2} m_2 a$$

$$m_3 g - T_2 = m_3 a$$

(a) The Acceleration. Add the equations, taking advantage of Newton's Third Law of Action and Reaction.

$$m_3 g = \frac{2}{5} m_1 a + \frac{1}{2} m_2 a + m_3 a$$

$$m_3 g = (\frac{2}{5} m_1 + \frac{1}{2} m_2 + m_3) a$$

$$10 m_3 g = (4m_1 + 5m_2 + 10m_3) a$$

$$a = (\frac{10m_3}{4m_1 + 5m_2 + 10m_3}) g$$

Dimensions look good and $\lim_{m_3 \to \infty} a = \lim_{m_3 \to \infty} (\frac{10m_3}{4m_1 + 5m_2 + 10m_3}) g = g$ as expected.

And if the sphere or disk mass goes to infinity, there is no acceleration.

(b) The Tensions.

$$T_1 = \frac{2}{5} m_1 a \quad => \quad T_1 = \frac{2}{5} m_1 (\frac{10m_3}{4m_1 + 5m_2 + 10m_3}) g \quad => \quad T_1 = (\frac{4m_1 m_3}{4m_1 + 5m_2 + 10m_3}) g$$

$$m_3 g - T_2 = m_3 a \quad => \quad T_2 = m_3 g - m_3 a \quad => \quad T_2 = m_3 (g - a)$$

$$T_2 = m_3 \left[g - (\frac{10m_3}{4m_1 + 5m_2 + 10m_3}) g \right] \quad => \quad T_2 = m_3 \left[1 - \frac{10m_3}{4m_1 + 5m_2 + 10m_3} \right] g$$

$$T_2 = m_3 \left[\frac{4m_1 + 5m_2 + 10m_3}{4m_1 + 5m_2 + 10m_3} - \frac{10m_3}{4m_1 + 5m_2 + 10m_3} \right] g$$

$$T_2 = m_3 \left[\frac{4m_1 + 5m_2}{4m_1 + 5m_2 + 10m_3} \right] g$$

$$T_2 = \left[\frac{(4m_1 + 5m_2) m_3}{4m_1 + 5m_2 + 10m_3} \right] g$$

We can do a check for the case where the sphere mass is zero as we did this problem in the previous section. We got the following.

$$a = \frac{2m}{2m + M} g \qquad T = \frac{mM}{2m + M} g$$

To relate to our more general problem here:

$$a = (\frac{10m_3}{4m_1 + 5m_2 + 10m_3}) g \qquad T_2 = \left[\frac{(4m_1 + 5m_2) m_3}{4m_1 + 5m_2 + 10m_3} \right] g$$

we take $m_1 = 0$, $m_2 = M$, and $m_3 = m$.

$$a = (\frac{10m_3}{4m_1 + 5m_2 + 10m_3}) g \to (\frac{10m}{4 \cdot 0 + 5M + 10m}) g = (\frac{10m}{10m + 5M}) g = \frac{2m}{2m + M} g$$

$$T_2 = \left[\frac{(4m_1 + 5m_2) m_3}{4m_1 + 5m_2 + 10m_3} \right] g \to \left[\frac{(4 \cdot 0 + 5M) m}{4 \cdot 0 + 5M + 10m} \right] g = \left[\frac{5Mm}{5M + 10m} \right] g = \frac{Mm}{M + 2m} g = \frac{mM}{2m + M} g$$

Everything checks out and there is no tension 1:

$$T_1 = (\frac{4m_1 m_3}{4m_1 + 5m_2 + 10m_3})g \to (\frac{4\cdot 0\cdot m}{4\cdot 0 + 5M + 10m})g = 0$$

(c) The Velocity.

$$K_1 + U_1 = K_2 + U_2$$

$$0 + m_3 gh = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}m_3 v^2 + 0$$

$$m_3 gh = \frac{1}{2}(\frac{2}{5}mr_1^2)\omega_1^2 + \frac{1}{2}(\frac{1}{2}m_2 r_2^2)\omega_1^2 + \frac{1}{2}m_3 v^2$$

$$m_3 gh = \frac{1}{2}(\frac{2}{5}mr_1^2)(\frac{v}{r_1})^2 + \frac{1}{2}(\frac{1}{2}m_2 r_2^2)(\frac{v}{r_2})^2 + \frac{1}{2}m_3 v^2$$

$$m_3 gh = \frac{1}{5}mv^2 + \frac{1}{4}m_2 v^2 + \frac{1}{2}m_3 v^2$$

$$20m_3 gh = 4m_1 v^2 + 5m_2 v^2 + 10m_3 v^2$$

$$20m_3 gh = (4m_1 + 5m_2 + 10m_3)v^2$$

$$v = \sqrt{(\frac{20m_3}{4m_1 + 5m_2 + 10m_3})gh}$$

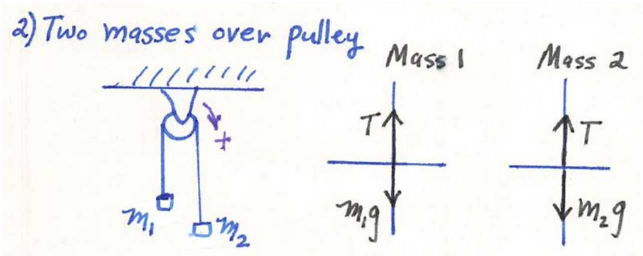
We can check this formula against $v = \sqrt{\frac{4mgh}{2m+M}}$ with $m_1 = 0$, $m_2 = M$, and $m_3 = m$.

$$v = \sqrt{(\frac{20m_3}{4m_1 + 5m_2 + 10m_3})gh} \to \sqrt{(\frac{20m}{4\cdot 0 + 5M + 10m})gh} = \sqrt{(\frac{20m}{10m + 5M})gh} = \sqrt{\frac{4mgh}{2m+M}}$$

Our original problem in this section is frequently given where the sphere is replaced with a spherical shell. Can you trace through the problem with

$$I_1 = \frac{2}{5}m_1 r_1^2 \text{ replaced with } I_1 = \frac{2}{3}m_1 r_1^2 ?$$

M6. The Atwood Machine. The Atwood machine is names after its inventor George Atwood (1745-1807), a mathematician, who used his invention to study Newton's Law of motion. We will check out his machine in this section. In fact, we have already looked at such an arrangement. Below is the figure we encountered in Chapter E.

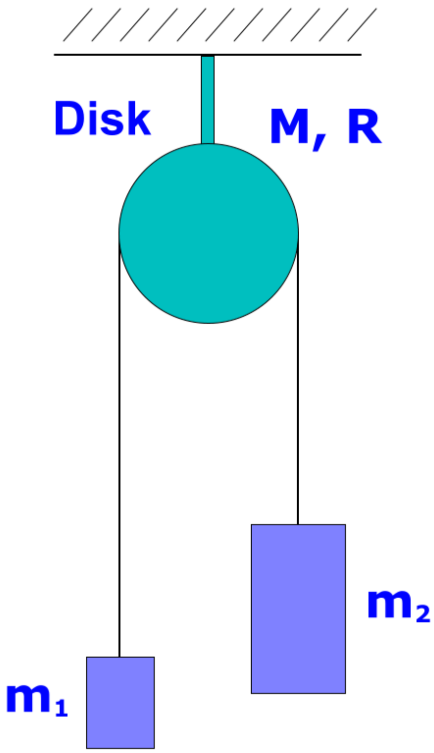


We found the following results.

$$a = (\frac{m_2 - m_1}{m_1 + m_2})g$$

$$T = (\frac{2m_1 m_2}{m_1 + m_2})g$$

These formulas apply to the Atwood machine with a massless pulley or a pulley over which the rope slides with no friction.



The Atwood machine is shown in the left figure.

The pulley is a disk. The mass is M and the radius is R.

Therefore, the moment of inertia is

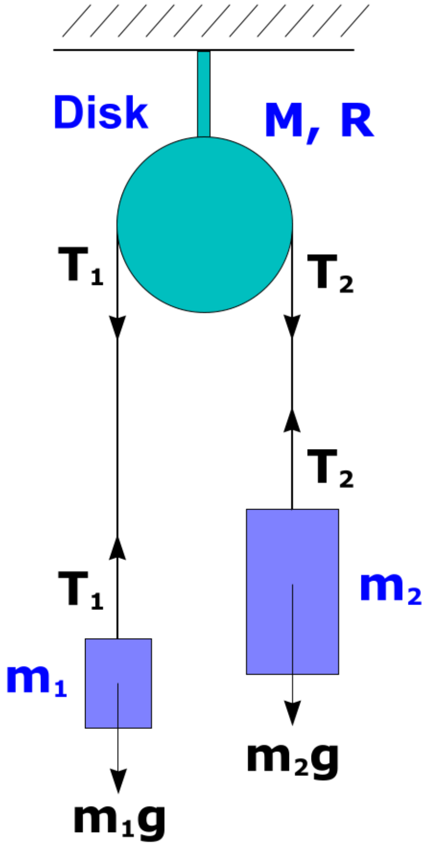
$$I = \frac{1}{2}MR^2 .$$

Atwood's intent was to use heavy weights so that he had a massless pulley. In that sense, we already have done the problem for the Atwood machine.

However, a real Atwood machine will have some rotational inertia in the pulley.

We proceed to analyze a realistic Atwood machine where the pulley has rotational inertia and the friction between the rope and pulley causes the pulley to rotate.

Find (a) the acceleration and (b) the tensions in the rope.



We have three objects. There will be three equations. We choose the positive direction to be mass m_2 moving down. Consistent with this choice, positive rotation for the pulley is clockwise and positive for m_1 is moving upward.

$$F_1 = m_1 a \quad => \quad T_1 - m_1 g = m_1 a$$

$$\tau = I\alpha \quad => \quad T_2 R - T_1 R = \frac{1}{2} M R^2 \frac{a}{R}$$

$$F_2 = m_2 a \quad => \quad m_2 g - T_2 = m_2 a$$

The three equations, with slight tidying up, are

$$T_1 - m_1 g = m_1 a \, ,$$

$$T_2 - T_1 = \frac{1}{2} M a \, ,$$

$$m_2 g - T_2 = m_2 a \, .$$

(a) The Acceleration. We add the equations, taking advantage of Newton's Third Law of Action and Reaction.

$$\begin{aligned} T_1 - m_1 g &= m_1 a \\ T_2 - T_1 &= \frac{1}{2} M a \\ m_2 g - T_2 &= m_2 a \end{aligned}$$

The result of adding the three equations is

$$m_2 g - m_1 g = m_1 a + \frac{1}{2} M a + m_2 a \, .$$

$$(m_2 - m_1) g = (m_1 + \frac{1}{2} M + m_2) a$$

$$a = \frac{m_2 - m_1}{m_1 + \frac{1}{2} M + m_2} g$$

$$a = \frac{2(m_2 - m_1)}{2(m_1 + m_2) + M} g$$

(b) The Tensions.

$$T_1 - m_1 g = m_1 a \quad \Rightarrow \quad T_1 = m_1 a + m_1 g \quad \Rightarrow \quad T_1 = m_1(a+g)$$

$$a + g = \frac{2(m_2 - m_1)}{2(m_1 + m_2) + M} g + g$$

$$\frac{a+g}{g} = \frac{2(m_2 - m_1)}{2(m_1 + m_2) + M} + 1$$

$$\frac{a+g}{g} = \frac{2(m_2 - m_1)}{2(m_1 + m_2) + M} + \frac{2(m_1 + m_2) + M}{2(m_1 + m_2) + M}$$

$$\frac{a+g}{g} = \frac{2(m_2 - m_1) + 2(m_1 + m_2) + M}{2(m_1 + m_2) + M}$$

$$\frac{a+g}{g} = \frac{2m_2 - 2m_1 + 2m_1 + 2m_2 + M}{2(m_1 + m_2) + M}$$

$$\frac{a+g}{g} = \frac{4m_2 + M}{2(m_1 + m_2) + M} \quad \Rightarrow \quad a + g = \frac{4m_2 + M}{2(m_1 + m_2) + M} g$$

$$T_1 = m_1(a+g) \quad \Rightarrow \quad T_1 = \frac{m_1(4m_2 + M)}{2(m_1 + m_2) + M} g$$

$$m_2 g - T_2 = m_2 a \quad \Rightarrow \quad T_2 = m_2 g - m_2 a \quad \Rightarrow \quad T_2 = m_2(g - a)$$

$$a = \frac{2(m_2 - m_1)}{2(m_1 + m_2) + M} g \quad \Rightarrow \quad g - a = g - \frac{2(m_2 - m_1)}{2(m_1 + m_2) + M} g \quad \Rightarrow \quad \frac{g-a}{g} = 1 - \frac{2(m_2 - m_1)}{2(m_1 + m_2) + M}$$

$$\frac{g-a}{g} = \frac{2(m_1 + m_2) + M}{2(m_1 + m_2) + M} - \frac{2(m_2 - m_1)}{2(m_1 + m_2) + M} \quad \Rightarrow \quad \frac{g-a}{g} = \frac{2m_1 + 2m_2 + M - 2m_2 + 2m_1)}{2(m_1 + m_2) + M}$$

$$\frac{g-a}{g} = \frac{4m_1 + M}{2(m_1 + m_2) + M} \quad \Rightarrow \quad g - a = \frac{4m_1 + M}{2(m_1 + m_2) + M} g$$

$$T_2 = m_2(g - a)$$

$$T_2 = \frac{m_2(4m_1 + M)}{2(m_1 + m_2) + M} g$$

Summary: $a = \frac{2(m_2 - m_1)}{2(m_1 + m_2) + M} g$ $\quad T_1 = \frac{m_1(4m_2 + M)}{2(m_1 + m_2) + M} g$ $\quad T_2 = \frac{m_2(4m_1 + M)}{2(m_1 + m_2) + M} g$

We now check if we get the expected for the massless pulley. $a = (\frac{m_2 - m_1}{m_1 + m_2}) g$ $\quad T = (\frac{2m_1 m_2}{m_1 + m_2}) g$

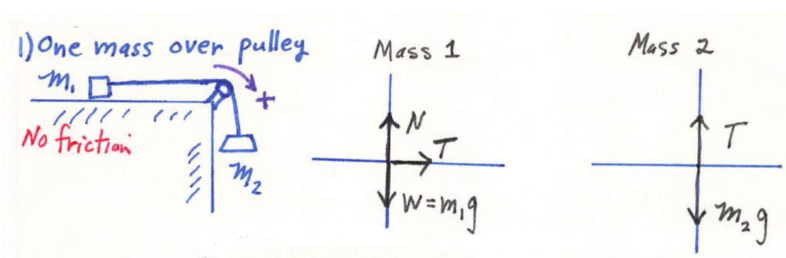
We use our most general formulas with $M = 0$. We should also find $T_1 = T_2$.

$$a = \frac{2(m_2 - m_1)}{2(m_1 + m_2) + M} g \to \frac{2(m_2 - m_1)}{2(m_1 + m_2)} g = \frac{m_2 - m_1}{m_1 + m_2} g$$

$$T_1 = \frac{m_1(4m_2 + M)}{2(m_1 + m_2) + M} g \to \frac{m_1(4m_2)}{2(m_1 + m_2)} g = \frac{2m_1 m_2}{m_1 + m_2} g$$

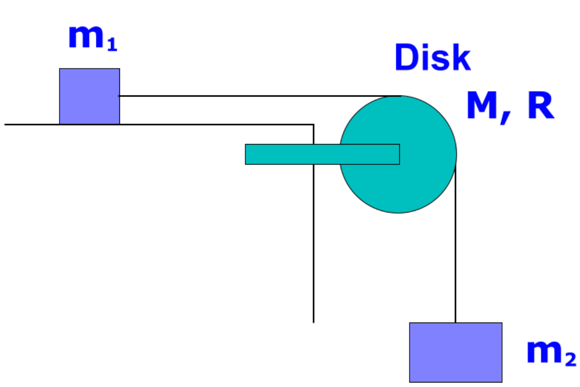
$$T_2 = \frac{m_2(4m_1 + M)}{2(m_1 + m_2) + M} g \to \frac{m_2(4m_1)}{2(m_1 + m_2)} g = \frac{2m_1 m_2}{m_1 + m_2} g$$

M8. Pulling Mass in Table. We return to our earlier pulley and table problem, but now with rotational inertia for the pulley. We found in Chapter E the following solution for the massless pulley or rope sliding over a pulley with no friction.



$$a = (\frac{m_2}{m_1 + m_2}) g$$

$$T = (\frac{m_1 m_2}{m_1 + m_2}) g$$

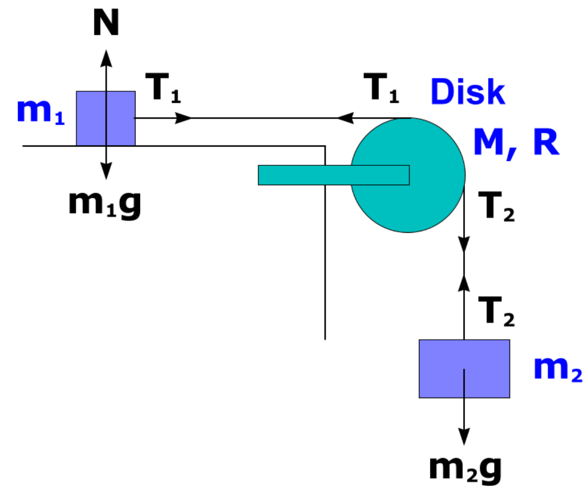


The arrangement at the left includes rotational inertial for the pulley disk but there is no friction for the mass on the table.

(a) Find the acceleration.

(b) Find the tension in each section of the rope or cable.

(c) Check that when pulley is massless that you recover the above formulas from before.



There are three objects to consider.

$$F_1 = m_1 a \quad \text{=>} \quad T_1 = m_1 a$$

$$\tau = I\alpha \quad \text{=>} \quad T_2 R - T_1 R = \frac{1}{2} M R^2 \frac{a}{R}$$

$$F_2 = m_2 a \quad \text{=>} \quad m_2 g - T_2 = m_2 a$$

The three equations, with slight tidying up, are

$$T_1 = m_1 a \, ,$$
$$T_2 - T_1 = \frac{1}{2} M a \, ,$$
$$m_2 g - T_2 = m_2 a \, .$$

(a) The Acceleration. Add the equations.

$$m_2 g = m_1 a + \frac{1}{2} M a + m_2 a$$

$$m_2 g = (m_1 + \frac{1}{2} M + m_2) a \quad \text{=>} \quad a = \frac{m_2}{m_1 + \frac{1}{2} M + m_2} g \quad \text{=>} \quad a = \frac{2m_2}{2m_1 + M + 2m_2} g$$

$$a = \frac{2m_2}{2(m_1 + m_2) + M} g$$

(b) The Tensions.

$$T_1 = m_1 a$$

$$T_1 = \frac{2m_1 m_2}{2(m_1 + m_2) + M} g$$

$$m_2 g - T_2 = m_2 a \quad \text{=>} \quad m_2 g - m_2 a = T_2 \quad \text{=>} \quad T_2 = m_2 (g - a)$$

$$g - a = g - \frac{2m_2}{2(m_1 + m_2) + M} g \quad \text{=>} \quad \frac{g - a}{g} = 1 - \frac{2m_2}{2(m_1 + m_2) + M}$$

$$\frac{g-a}{g}=\frac{2(m_1+m_2)+M}{2(m_1+m_2)+M}-\frac{2m_2}{2(m_1+m_2)+M}$$

$$\frac{g-a}{g}=\frac{2m_1+2m_2+M-2m_2}{2(m_1+m_2)+M} \quad => \quad \frac{g-a}{g}=\frac{2m_1+M}{2(m_1+m_2)+M} \quad => \quad g-a=\frac{2m_1+M}{2(m_1+m_2)+M}g$$

$$T_2=m_2(g-a) \quad => \quad T_2=\frac{(2m_1+M)m_2}{2(m_1+m_2)+M}g$$

(c) Correspondence Formulas Reducing to Frictionless Pulley. Take $M=0$ and see if we get

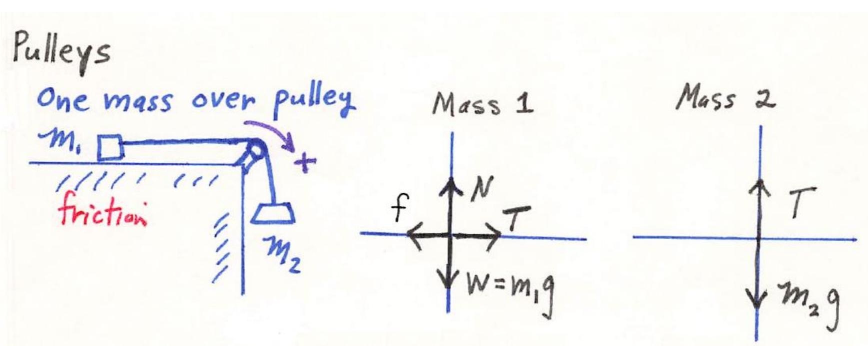
$$a=(\frac{m_2}{m_1+m_2})g \quad \text{and} \quad T=(\frac{m_1m_2}{m_1+m_2})g\ .$$

$$a=\frac{2m_2}{2(m_1+m_2)+M}g \rightarrow \frac{2m_2}{2(m_1+m_2)}g=(\frac{m_2}{m_1+m_2})g$$

$$T_1=\frac{2m_1m_2}{2(m_1+m_2)+M}g \rightarrow \frac{2m_1m_2}{2(m_1+m_2)}g=\frac{m_1m_2}{m_1+m_2}g$$

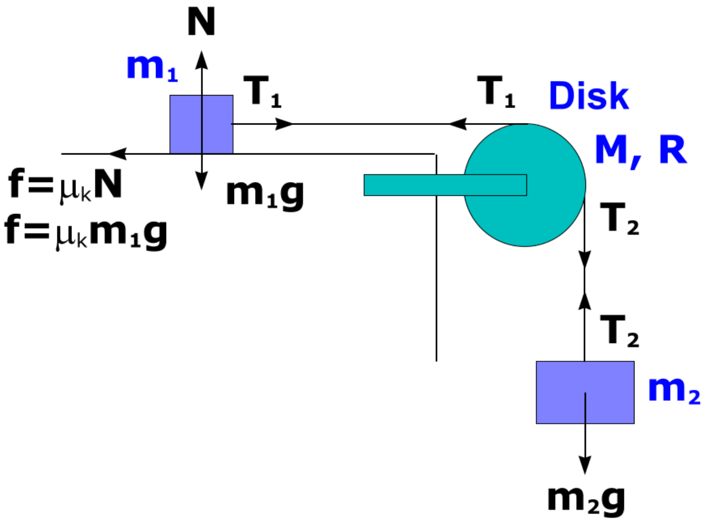
$$T_2=\frac{(2m_1+M)m_2}{2(m_1+m_2)+M}g \rightarrow \frac{2m_1m_2}{2(m_1+m_2)}g=\frac{m_1m_2}{m_1+m_2}g$$

What about adding friction for the mass on the table. In Chapter E we found for a massless pulley the following formulas.



$$a=(\frac{m_2-\mu_k m_1}{m_1+m_2})g$$

$$T=(\frac{m_1m_2}{m_1+m_2})(1+\mu_k)g$$



Our three equations are now as follows.

$$F_1 = m_1 a \quad => \quad T_1 - \mu_k m_1 g = m_1 a$$

$$\tau = I\alpha \quad => \quad T_2 R - T_1 R = \frac{1}{2} M R^2 \frac{a}{R}$$

$$F_2 = m_2 a \quad => \quad m_2 g - T_2 = m_2 a$$

The three equations, with slight tidying up the second one, are

$$T_1 - \mu_k m_1 g = m_1 a\,,$$
$$T_2 - T_1 = \frac{1}{2} M a\,,$$
$$m_2 g - T_2 = m_2 a\,.$$

Add the equations.

$$(m_2 - \mu_k m_1) g = m_1 a + \frac{1}{2} M a + m_2 a \quad => \quad (m_2 - \mu_k m_1) g = (m_1 + \frac{1}{2} M + m_2) a$$

$$a = \frac{m_2 - \mu_k m_1}{m_1 + \frac{1}{2} M + m_2} g \quad => \quad a = \frac{m_2 - \mu_k m_1}{m_1 + m_2 + M/2} g$$

When $M \to 0$ we recover the result from Chapter E above.

$$T_1 - \mu_k m_1 g = m_1 a \quad => \quad T_1 = m_1 a + \mu_k m_1 g \quad => \quad T_1 = m_1 (a + \mu_k g)$$

$$T_1 = m_1 (\frac{m_2 - \mu_k m_1}{m_1 + m_2 + M/2} g + \mu_k g) \quad => \quad T_1 = m_1 (\frac{m_2 - \mu_k m_1}{m_1 + m_2 + M/2} + \mu_k) g$$

$$T_1 = m_1 (\frac{m_2 - \mu_k m_1}{m_1 + m_2 + M/2} + \frac{m_1 + m_2 + M/2}{m_1 + m_2 + M/2} \mu_k) g$$

$$T_1 = m_1 (\frac{m_2 - \mu_k m_1 + \mu_k m_1 + \mu_k m_2 + \mu_k M/2}{m_1 + m_2 + M/2}) g$$

$$T_1 = m_1 (\frac{m_2 + \mu_k m_2 + \mu_k M/2}{m_1 + m_2 + M/2}) g$$

$$T_1 = m_1 \left[\frac{m_2(1+\mu_k) + \mu_k M/2}{m_1 + m_2 + M/2} \right] g$$

$$m_2 g - T_2 = m_2 a \quad => \quad T_2 = m_2 g - m_2 a \quad => \quad T_2 = m_2(g-a)$$

$$T_2 = m_2(g-a)$$

$$T_2 = m_2(g - \frac{m_2 - \mu_k m_1}{m_1 + m_2 + M/2} g) \quad => \quad T_2 = m_2(1 - \frac{m_2 - \mu_k m_1}{m_1 + m_2 + M/2})g$$

$$T_2 = m_2(\frac{m_1 + m_2 + M/2}{m_1 + m_2 + M/2} - \frac{m_2 - \mu_k m_1}{m_1 + m_2 + M/2})g$$

$$T_2 = m_2(\frac{m_1 + m_2 + M/2 - m_2 + \mu_k m_1}{m_1 + m_2 + M/2})g$$

$$T_2 = m_2(\frac{m_1 + M/2 + \mu_k m_1}{m_1 + m_2 + M/2})g$$

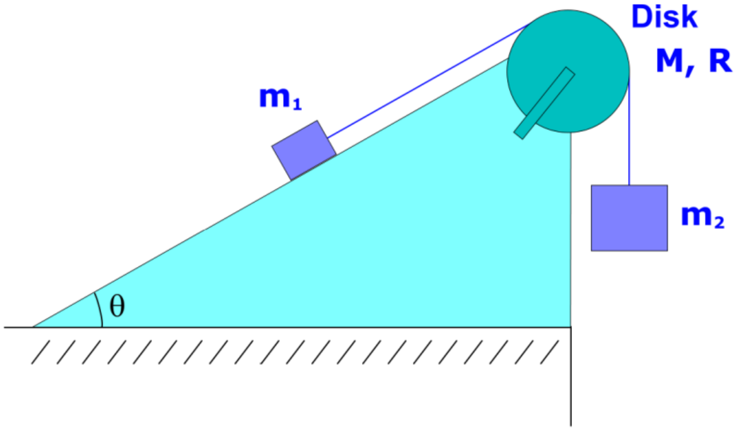
$$T_2 = m_2 \left[\frac{m_1(1+\mu_k) + M/2}{m_1 + m_2 + M/2} \right] g$$

Do we get $T_1 = T_2 = T = (\frac{m_1 m_2}{m_1 + m_2})(1+\mu_k)g$ when $M \to 0$?

$$T_1 = m_1 \left[\frac{m_2(1+\mu_k) + \mu_k M/2}{m_1 + m_2 + M/2} \right] g \to m_1 \left[\frac{m_2(1+\mu_k)}{m_1 + m_2} \right] g = \frac{m_1 m_2}{m_1 + m_2}(1+\mu_k)$$

$$T_2 = m_2 \left[\frac{m_1(1+\mu_k) + M/2}{m_1 + m_2 + M/2} \right] g \to m_2 \left[\frac{m_1(1+\mu_k)}{m_1 + m_2} \right] g = \frac{m_1 m_2}{m_1 + m_2}(1+\mu_k)$$

M9. Pulling Mass Up an Incline. We did the hanging mass pulling up a mass on an incline both with friction on the incline and a frictionless incline. But we had a massless pulley or one where the rope slide over the pulley with no rotation.

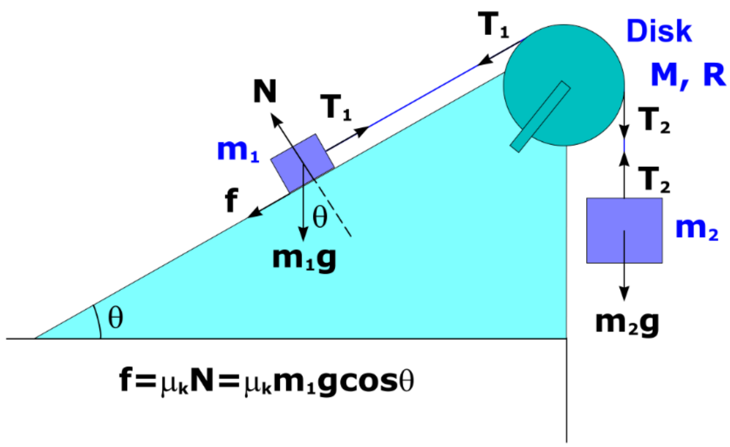


Pulling a mass up an incline with a massless pulley led to these equations derived in Chapter E.

$$a = \frac{[m_2 - m_1(\sin\theta + \mu_k \cos\theta)]}{m_1 + m_2} g$$

$$T = (\frac{m_1 m_2}{m_1 + m_2})(1 + \sin\theta + \mu_k \cos\theta) g$$

These reduced to the first formulas we derived for a frictionless incline where $\mu_k = 0$. In this section we extend the above boxed formulas to include a disk pulley with rotational inertia.



Let's begin.

$$F_1 = m_1 a \ \ =>$$

$$T_1 - f - m_1 g \sin\theta = m_1 a$$

$$\tau = I\alpha \quad => \quad T_2 R - T_1 R = \frac{1}{2} M R^2 \frac{a}{R}$$

$$F_2 = m_2 a \quad => \quad m_2 g - T_2 = m_2 a$$

The frictional force $f = \mu_k N$ and the normal force $N = m_1 g \cos\theta$ from the forces perpendicular to the incline. The frictional force is $f = \mu_k N = \mu_k m_1 g \cos\theta$. It is important to stress that we are doing the problem for the mass going up the incline. In this situation the frictional force is done the incline, always opposite to your direction of motion.

The three equations, with substituting the frictional force in the first equation and tidying up the second equation are.

$$T_1 - \mu_k m_1 g \cos\theta - m_1 g \sin\theta = m_1 a$$

$$T_2 - T_1 = \frac{1}{2} M a$$

$$m_2 g - T_2 = m_2 a \, .$$

Add the equations.

$$m_2 g - \mu_k m_1 g \cos\theta - m_1 g \sin\theta = m_1 a + (M/2) a + m_2 a$$

$$m_2 g - \mu_k m_1 g\cos\theta - m_1 g\sin\theta = (m_1 + \frac{1}{2}M + m_2)a$$

$$a = (\frac{m_2 - \mu_k m_1\cos\theta - m_1\sin\theta}{m_1 + \frac{1}{2}M + m_2})g \quad => \quad a = \left[\frac{m_2 - m_1(\mu_k\cos\theta + \sin\theta)}{m_1 + m_2 + M/2}\right]g$$

$$a = \left[\frac{m_2 - m_1(\sin\theta + \mu_k\cos\theta)}{m_1 + m_2 + M/2}\right]g$$

$$T_1 - \mu_k m_1 g\cos\theta - m_1 g\sin\theta = m_1 a \quad => \quad T_1 = m_1 a + \mu_k m_1 g\cos\theta + m_1 g\sin\theta$$

$$T_1 = m_1(a + \mu_k g\cos\theta + g\sin\theta) \quad => \quad T_1 = m_1(a + g\sin\theta + \mu_k g\cos\theta)$$

$$T_1 = m_1\left[\frac{m_2 - m_1(\sin\theta + \mu_k\cos\theta)}{m_1 + m_2 + M/2}g + g\sin\theta + \mu_k g\cos\theta\right]$$

$$T_1 = m_1\left[\frac{m_2 - m_1(\sin\theta + \mu_k\cos\theta)}{m_1 + m_2 + M/2} + (\sin\theta + \mu_k\cos\theta)\right]g$$

$$T_1 = m_1\left[\frac{m_2 - m_1(\sin\theta + \mu_k\cos\theta)}{m_1 + m_2 + M/2} + \frac{m_1 + m_2 + M/2}{m_1 + m_2 + M/2}(\sin\theta + \mu_k\cos\theta)\right]g$$

$$T_1 = m_1\left[\frac{m_2 - m_1(\sin\theta + \mu_k\cos\theta) + m_1(\sin\theta + \mu_k\cos\theta) + m_2(\sin\theta + \mu_k\cos\theta) + M(\sin\theta + \mu_k\cos\theta)/2}{m_1 + m_2 + M/2}\right]g$$

$$T_1 = m_1\left[\frac{m_2 + m_2(\sin\theta + \mu_k\cos\theta) + M(\sin\theta + \mu_k\cos\theta)/2}{m_1 + m_2 + M/2}\right]g$$

$$T_1 = m_1\left[\frac{m_2 + (m_2 + M/2)(\sin\theta + \mu_k\cos\theta)}{m_1 + m_2 + M/2}\right]g$$

$$m_2 g - T_2 = m_2 a \quad => \quad m_2 g = m_2 a + T_2 \quad => \quad T_2 = m_2 g - m_2 a \quad => \quad T_2 = m_2(g - a)$$

$$T_2 = m_2\left[g - \frac{m_2 - m_1(\sin\theta + \mu_k\cos\theta)}{m_1 + m_2 + M/2}g\right] \quad => \quad T_2 = m_2\left[1 - \frac{m_2 - m_1(\sin\theta + \mu_k\cos\theta)}{m_1 + m_2 + M/2}\right]g$$

$$T_2 = m_2\left[\frac{m_1+m_2+M/2}{m_1+m_2+M/2} - \frac{m_2-m_1(\sin\theta+\mu_k\cos\theta)}{m_1+m_2+M/2}\right]g$$

$$T_2 = m_2\left[\frac{m_1+m_2+M/2-m_2+m_1(\sin\theta+\mu_k\cos\theta)}{m_1+m_2+M/2}\right]g$$

$$T_2 = m_2\left[\frac{m_1+M/2+m_1(\sin\theta+\mu_k\cos\theta)}{m_1+m_2+M/2}\right]g$$

$$T_2 = m_2\left[\frac{m_1(1+\sin\theta+\mu_k\cos\theta)+M/2}{m_1+m_2+M/2}\right]g$$

Summary: $a = \left[\dfrac{m_2-m_1(\sin\theta+\mu_k\cos\theta)}{m_1+m_2+M/2}\right]g$

$$T_1 = m_1\left[\frac{m_2+(m_2+M/2)(\sin\theta+\mu_k\cos\theta)}{m_1+m_2+M/2}\right]g$$

$$T_2 = m_2\left[\frac{m_1(1+\sin\theta+\mu_k\cos\theta)+M/2}{m_1+m_2+M/2}\right]g$$

Let $M \to 0$ and see if these equations reduce to

$$a = \frac{[m_2-m_1(\sin\theta+\mu_k\cos\theta)]}{m_1+m_2}g$$

$$T = \left(\frac{m_1m_2}{m_1+m_2}\right)(1+\sin\theta+\mu_k\cos\theta)g$$

$$a = \left[\frac{m_2-m_1(\sin\theta+\mu_k\cos\theta)}{m_1+m_2+M/2}\right]g \to \frac{[m_2-m_1(\sin\theta+\mu_k\cos\theta)]}{m_1+m_2}g$$

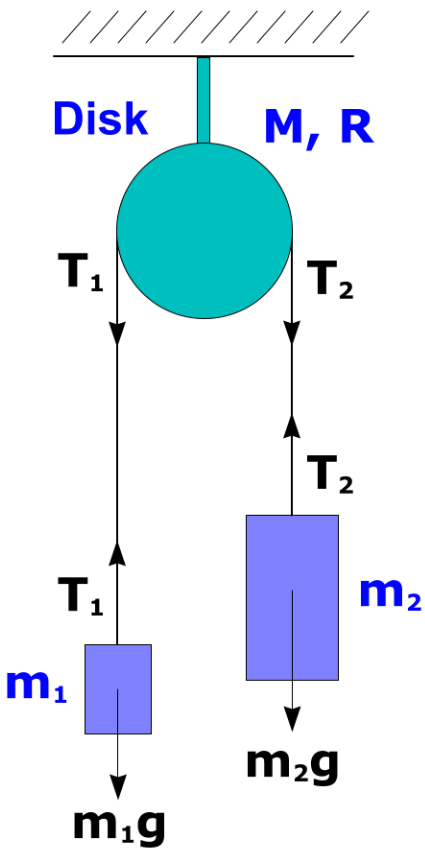
$$T_1 = m_1\left[\frac{m_2+(m_2+M/2)(\sin\theta+\mu_k\cos\theta)}{m_1+m_2+M/2}\right]g \to m_1\left[\frac{m_2+m_2(\sin\theta+\mu_k\cos\theta)}{m_1+m_2}\right]g$$

$$T_1 \to m_1\left[\frac{m_2(1+(\sin\theta+\mu_k\cos\theta)}{m_1+m_2}\right]g = (\frac{m_1 m_2}{m_1+m_2})(1+\sin\theta+\mu_k\cos\theta)g = T$$

$$T_2 = m_2\left[\frac{m_1(1+\sin\theta+\mu_k\cos\theta)+M/2}{m_1+m_2+M/2}\right]g \to m_2\left[\frac{m_1(1+\sin\theta+\mu_k\cos\theta)}{m_1+m_2}\right]$$

$$T_2 = \to (\frac{m_1 m_2}{m_1+m_2})(1+\sin\theta+\mu_k\cos\theta)g = T$$

M10. Belt Friction. The friction between the cable and the pulley varies over the pulley. This variation must be present since the tensions at the extreme points of contact are not the same. See the figure of the Atwood machine we did earlier.



Summary: $$a = \frac{2(m_2 - m_1)}{2(m_1+m_2)+M}g$$

$$T_1 = \frac{m_1(4m_2+M)}{2(m_1+m_2)+M}g \qquad T_2 = \frac{m_2(4m_1+M)}{2(m_1+m_2)+M}g$$

Since $m_2 > m_1$ for our arrangement at the left,

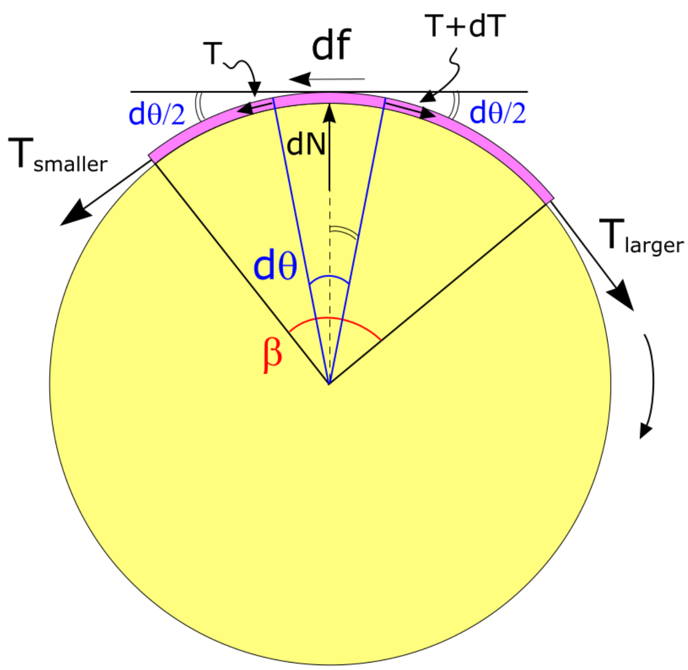
$$T_2 > T_1.$$

What minimum coefficient of static friction do we need between the belt and the pulley so that there is no slipping? For a specific case to analyze, take $m_1 = 2M$ and $m_2 = 5M$.

$$T_1 = \frac{m_1(4m_2+M)}{2(m_1+m_2)+M}g = \frac{2M(4\cdot 5M+M)}{2(2M+5M)+M}g$$

$$T_1 = \frac{2(20+1)}{2(7)+1}Mg = \frac{2(21)}{15}Mg = \frac{2(7)}{5}Mg = \frac{14}{5}Mg$$

$$T_2 = \frac{m_2(4m_1+M)}{2(m_1+m_2)+M}g = \frac{5M(4\cdot 2M+M)}{15M}g = \frac{5(8+1)}{15}Mg = \frac{5(9)}{15}Mg = \frac{5(3)}{5}Mg = \frac{15}{5}Mg$$



A pulley with a belt is shown at the left. The larger tension T_2 is labeled as T_{larger} and the smaller tension T_1 is labeled T_{smaller} so that when you see the formula we drive here months later you will not have to remember if T_2 is larger than T_1 .

The angle swept out between T_{smaller} and T_{larger} is β. For the analysis, we consider a small angle $d\theta$ and a small segment of belt.

The belt has such a small mass compared to the pulley that we consider the belt massless. You will see shortly that we actually want to set the mass element to zero for another reason.

We apply Newton's Law to the belt element situated at the top of the pulley. We have an equation for the x-direction or tangential direction and one for the y-direction or normal axis. For the moment we include a nonzero mass element dm .

$$\sum F_x = (T + dT)\cos\frac{d\theta}{2} - T\cos\frac{d\theta}{2} - df = (dm)a$$

$$\sum F_y = dN - T\sin\frac{d\theta}{2} - (T + dT)\sin\frac{d\theta}{2} = 0$$

Immediate simplification leads to

$$\left\langle \begin{matrix} dT\cos\frac{d\theta}{2} - df = (dm)a \\ dN - 2T\sin\frac{d\theta}{2} - dT\sin\frac{d\theta}{2} = 0 \end{matrix} \right\rangle,$$

where I have bracketed the pair of equations to keep track of them. I learned this trick from my goo friend Paul Ottinger in graduate school. He went on to get his Ph.D. in plasma physics.

Since $df = \mu dN$, the pair can be written as

$$\left\langle \begin{array}{l} dT\cos\frac{d\theta}{2} - \mu dN = (dm)a \\ dN - 2T\sin\frac{d\theta}{2} - dT\sin\frac{d\theta}{2} = 0 \end{array} \right\rangle$$

Let's take a moment an inspect the first equation, solving for μ.

$$dT\cos\frac{d\theta}{2} - \mu dN = (dm)a \quad \text{=>} \quad dT\cos\frac{d\theta}{2} - (dm)a = \mu dN \quad \text{=>} \quad \mu dN = dT\cos\frac{d\theta}{2} - (dm)a$$

The dm causes a reduction in the coefficient of friction due to the minus sign. So when we neglect the mass since it is small, our resulting coefficient of friction will be slightly larger. This slight increase is good because we want to play it safe and rather have our specification for the coefficient of friction a tad higher. Safety is of utmost importance in engineering. The pair of equations neglecting the belt mass is then

$$\left\langle \begin{array}{l} dT\cos\frac{d\theta}{2} - \mu dN = 0 \\ dN - 2T\sin\frac{d\theta}{2} - dT\sin\frac{d\theta}{2} = 0 \end{array} \right\rangle$$

which is equivalent to

$$\left\langle \begin{array}{l} dT\cos\frac{d\theta}{2} = \mu dN \\ dN = 2T\sin\frac{d\theta}{2} + dT\sin\frac{d\theta}{2} \end{array} \right\rangle.$$

Now it is time to invoke the two following important approximations in physics:

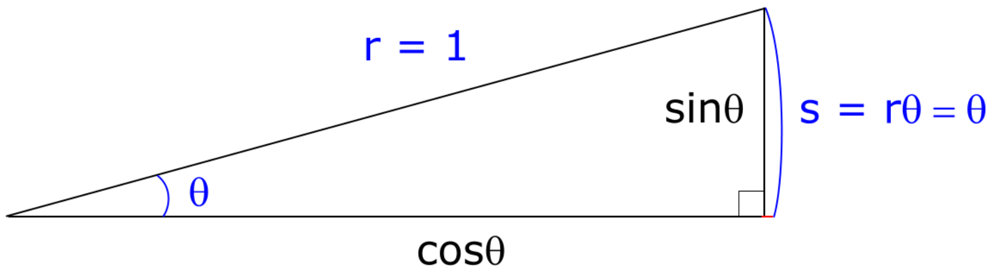
$\cos\theta \approx 1$ and $\sin\theta \approx \theta$,

when θ is very small and is an angle in radians.

The usual way to see these relations is to reflect back in your Calculus I class where you used differential calculus to express $\cos\theta$ and $\sin\theta$ as power series. Such power series are called Maclaurin series, the basic form for the more general Taylor series:

$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ...$ and $\sin\theta = \theta - \frac{\theta^3}{3!} - ...$, where θ is in radians.

But here is a cute geometric demonstration that will get us the first term in each.



When the angle θ is in radians, we can write the arc length as $s = r\theta$. Since $r = 1$, we readily find from the figure for small angles $\cos\theta \approx 1$ and $\sin\theta \approx \theta$. Check this approximation out for 10°.

$$\cos 10^\circ = 0.985 \approx 1$$

$$\sin 10^\circ = 0.174$$

$$10^\circ = 10^\circ \frac{\pi}{180^\circ} \text{radians} = 0.175 \approx \sin 10^\circ$$

And 10° is a not really a super small angle. The approximation still works!

Our equation pair with the approximation is then

$$\left\langle \begin{array}{l} dT \cos\dfrac{d\theta}{2} = \mu dN \\ dN = 2T \sin\dfrac{d\theta}{2} + dT \sin\dfrac{d\theta}{2} \end{array} \right\rangle \quad \Rightarrow \quad \left\langle \begin{array}{l} dT = \mu dN \\ dN = 2T\dfrac{d\theta}{2} + dT\dfrac{d\theta}{2} \end{array} \right\rangle.$$

At this point we can throw away the double differential since when we take the limit as our infinitesimals approach zero, a product of infinitesimals will run to zero faster. In other words, a differential is super small and a product of differentials can be discarded compared to a single differential. Our pair of equations become even simpler.

$$\left\langle \begin{array}{l} dT = \mu dN \\ dN = Td\theta \end{array} \right\rangle$$

We arrive at one equation by substituting $dN = Td\theta$ into $dT = \mu dN$

$$dT = \mu T d\theta .$$

This equation is a differential equation: $\dfrac{dT}{d\theta} = \mu T$.

We solve it by going back to our form $dT = \mu T d\theta$.

We separate the variable getting T on the left side and θ on the right.

The constant μ can go with the θ.

$$\frac{dT}{T} = \mu d\theta$$

We now integrate both sides, referring to our figure to get the limits.

$$\int_{T_{\text{smaller}}}^{T_{\text{larger}}} \frac{dT}{T} = \mu \int_0^{\beta} d\theta$$

$$\ln T \Big|_{T_{\text{smaller}}}^{T_{\text{larger}}} = \mu\, \theta \Big|_0^{\beta}$$

$$\ln T_{\text{larger}} - \ln T_{\text{smaller}} = \mu(\beta - 0)$$

$$\ln \frac{T_{\text{larger}}}{T_{\text{smaller}}} = \mu\beta$$

$$\frac{T_{\text{larger}}}{T_{\text{smaller}}} = e^{\mu\beta}$$

$$T_{\text{larger}} = T_{\text{smaller}} e^{\mu\beta}$$

For our pulley case under consideration $\beta = \pi$, $T_1 = \frac{14}{5} Mg$, and $T_2 = \frac{15}{5} Mg$.

$$\ln \frac{T_{\text{larger}}}{T_{\text{smaller}}} = \mu\beta = \mu\pi$$

$$\mu = \frac{1}{\pi} \ln \frac{T_{\text{larger}}}{T_{\text{smaller}}}$$

$$\mu = \frac{1}{\pi} \ln \frac{T_2}{T_1} = \frac{1}{\pi} \ln \frac{(15/5)Mg}{(14/5)Mg} = \frac{1}{\pi} \ln \frac{15}{14} = \frac{0.06899}{\pi} = 0.02196 = 0.02$$

This coefficient of static friction is easy to achieve.