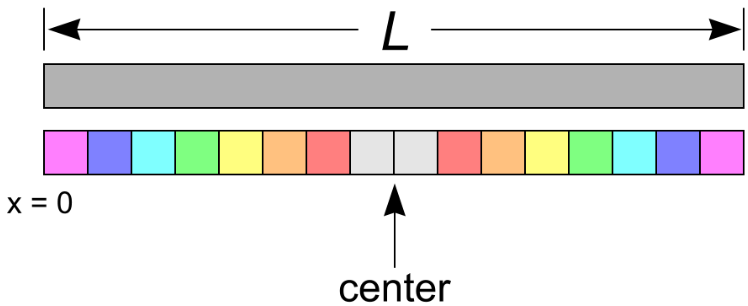
Physics I with Calculus, Prof. Ruiz (Doc), UNC-Asheville (1978-2021), DoctorPhys on YouTube
Chapter O. Rotational Statics. Prerequisite: Calculus I. Corequisite: Calculus II.

O0. The Center of Mass Review. The center of mass will be important in equilibrium problems. When you have an extended body, you can take gravity as acting at the center of mass.

Recall our calculation of the center of mass for a rod. We used the basic formula for the center of mass of two equal masses to give the midpoint between the two masses.

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m x_1 + m x_2}{m + m} = \frac{m(x_1 + x_2)}{2m} = \frac{x_1 + x_2}{2}$$



Then we broke up our rod into segments and paired them off, each one on the left with its partner on the right (by color).

The center of mass for each pair is at the midpoint between each.

Therefore, the center of mass for the entire rod is at the midpoint of the rod, which midpoint serves as the midpoint for all the mass segment pairs.

$$\boxed{x_{cm} = \frac{L}{2}}$$

Now let's relate all this to calculating torque due to gravity. Breaking the rod up into small slices, we sum the torques on the mass elements.

$$\tau = \sum_i (m_i g) x_i$$

But since g is constant,

$$\tau = (\sum_i m_i x_i) g \,.$$

However, $x_{cm} = \dfrac{\sum_i m_i x_i}{\sum_i m_i} = \dfrac{\sum_i m_i x_i}{M}$, which means $\sum_i m_i x_i = M x_{cm}$

Therefore, $\tau = (\sum_i m_i x_i) g = M x_{cm} g = M g x_{cm}$. You place the gravity force at the center of mass.

Technically, we should call this x_{cm} the center of gravity x_{cg} even though $x_{cm} = x_{cg}$.

The center of gravity is the balance point. If you hold up the mass there, the mass will not rotate. This equality is not always true. Suppose, the rod is large so that gravity varies some from place to place. Then, suppose the mass is not constant. Things can get confusing pretty fast. So we define three centers as follows. **But don't worry, in our course, dealing with objects on a celestial body, all three are the same – since we consider uniform densities and gravity is constant over the local region of our objects!**

1. The Centroid – the Geometric Center. This center can be defined using physics by assuming a uniform mass density. The subscript c indicates centroid.

$$x_c = \frac{\sum_i m_i x_i}{\sum_i m_i} \to \frac{\int x dm}{\int dm} = \frac{\int x dm}{M}$$

We are considering one dimension here. You would have a similar equation for the other two dimensions y and z. In one dimension $dm = \lambda dx$ and for a uniform mass density, λ is a constant. It is a linear density due to the one-dimensional characteristic of our problem. Let the length of the mass distribution be L with total mass M . Then $\lambda = \frac{M}{L}$ as we have done often earlier in our course.

$$x_c = \frac{\int x dm}{M} = \frac{1}{M}\int_0^L x\lambda dx = \frac{\lambda}{M}\int_0^L x dx = \frac{\lambda}{M}\frac{x^2}{2}\bigg|_0^L = \lambda\frac{1}{M}\frac{L^2}{2} = \frac{M}{L}\frac{1}{M}\frac{L^2}{2} = \frac{L}{2}$$

The physics disappears. The final answer is purely geometric: the center of the rod.

2. The Center of Mass. Here we must allow for nonuniform mass density.

$$x_{cm} = \frac{\int x dm}{M} = \frac{\int x\lambda(x)dx}{M}$$

3. The Center of Gravity. Now we allow gravity to vary from place to place under the rod and we are calculating the ratio of the torque to the weight:

$$x_{cm} = \frac{\int x(dm)g(x)}{\int (dm)g(x)} = \frac{\int x\lambda(x)g(x)dx}{\int g(x)\lambda(x)dx} = \frac{\int x\lambda(x)g(x)dx}{W}$$, where W is the weight.

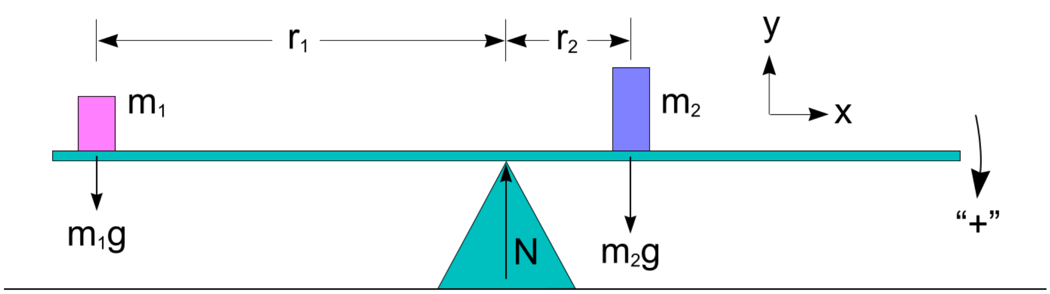
O1. The Seesaw. The seesaw where you can sit anywhere along the board provides us with a classic example of torque and static equilibrium. In the photo, dad needs to be closer to the fulcrum in the center to balance the smaller mass child farther away.

Courtesy Rachel Coleman, flickr, Seesaw at Évelle, Baubigny, France. [Attribution-ShareAlike 2.0]

See the force diagram for the board below, where we are neglecting the weight of the board for the moment. The force and torque equilibrium equations are below.

$$\sum F_y = N - m_1 g - m_2 g = 0$$

$$\sum \tau = -m_1 g r_1 + m_2 g r_2 = 0$$



For the torque equation we have indicated that "+" is clockwise in the diagram. We are also **free to pick our reference for the torques**. We choose the seesaw center. The torque equation indicates that the boy has to be farther from the fulcrum compared to dad. The force equation gives

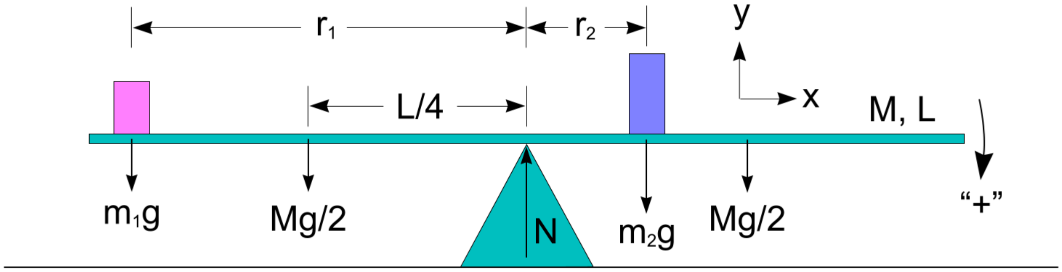
$$N = m_1 g + m_2 g \ .$$

The upward normal force on the board must support the two weights.

$$m_1 g r_1 = m_2 g r_2 \quad => \quad m_1 r_1 = m_2 r_2$$

$$r_1 = \frac{m_2}{m_1} r_2 > r_2$$

If dad is twice the weight of the son, then the son must be twice the distance away from the center, compared to the dad. What about not neglecting the mass of the board. See the figure below. The mass of the board is now taken to be M. The mass on the left side can be placed at the center of mass for the left side, i.e., a distance one quarter the board length to the left. A similar distance is used for the right side.



The equilibrium equations now become a little more complicated at first.

$$\begin{Bmatrix} \sum F_y = N - m_1 g - \dfrac{M}{2} g + m_2 g - \dfrac{M}{2} g = 0 \\ \sum \tau = -m_1 g r_1 - \dfrac{M}{2} g \dfrac{L}{4} + m_2 g r_2 + \dfrac{M}{2} g \dfrac{L}{4} = 0 \end{Bmatrix}$$

But we see immediately that the board effects are important for the normal force, but not for the torque equation, where they cancel out. Our pair of equilibrium equations

$$\begin{Bmatrix} N = m_1 g + \dfrac{M}{2} g + m_2 g + \dfrac{M}{2} g \\ m_1 g r_1 + \dfrac{M}{2} g \dfrac{L}{4} = m_2 g r_2 + \dfrac{M}{2} g \dfrac{L}{4} \end{Bmatrix}$$

reduce to

$$\begin{Bmatrix} N = m_1 g + m_2 g + Mg \\ m_1 g r_1 = m_2 g r_2 \end{Bmatrix}.$$

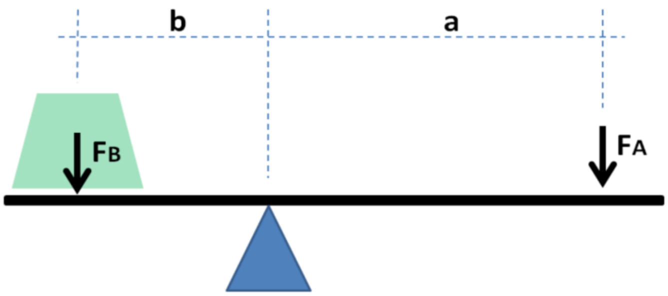
The fulcrum must now support the two weights on the board plus the weight of the board; however, the torque equation is the same as the one we had before, giving us

$$m_1 r_1 = m_2 r_2$$

and

$$r_1 = \frac{m_2}{m_1} r_2 > r_2 .$$

O2. The Lever and Mechanical Advantage. In the previous section we saw that a child can balance a parent on a seesaw provided that the child is farther away from the fulcrum compared to the parent. We say that the child has a mechanical advantage. See the lever below, which uses the same torque principle.



Courtesy Jimbowley
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Choosing the pivot at the fulcrum,

$$F_A a = F_B b$$

gives us equilibrium. The force F_A needed to hold up the weight F_B is less than the weight, i.e.,

$$F_A = F_B \frac{b}{a} < F_B$$

since $a > b$.

The mechanical advantage MA for the lever is the ratio F_B / F_A.

$$\boxed{MA = \frac{F_B}{F_A} = \frac{a}{b} > 1}$$

The principle of the lever has been known since ancient times. Archimedes was able to establish the above law for the lever c. 250 BCE.

Some texts call F_B the Load and F_A the Effort. $\boxed{\text{M.A.} = \frac{\text{Load}}{\text{Effort}} = \frac{\text{Output Force}}{\text{Input Force}}}$

Courtesy César Rincón
CR, Spanish Wikipedia
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The mechanical advantage for the lever at the left is 100/5 = 20. You can even count the notches. The 100-kg mass is one notch to the left and the 5-kg mass is 20 notches to the right.

Problem. The lady below exerts an effort force on a lever arm to lift a load weighing 420.0 newtons (94 pounds). Neglect the curve in the lever arm at the load. What must her downward force be to support the load when the lever arm is horizontal? The distance from the effort to the pivot is 3.0 meters (9.8 feet) and the distance from the pivot to the load is 0.5 meter (1.6 feet).

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Solution. The ratio of the longer distance along the lever arm to the shorter distance, each measured from the pivot, is the mechanical advantage

$$\text{M.A.} = \frac{r_{greater}}{r_{lesser}} = \frac{3.0 \text{ m}}{0.5 \text{ m}} = 6 .$$

Therefore, the effort needed to lift the 150. newtons is

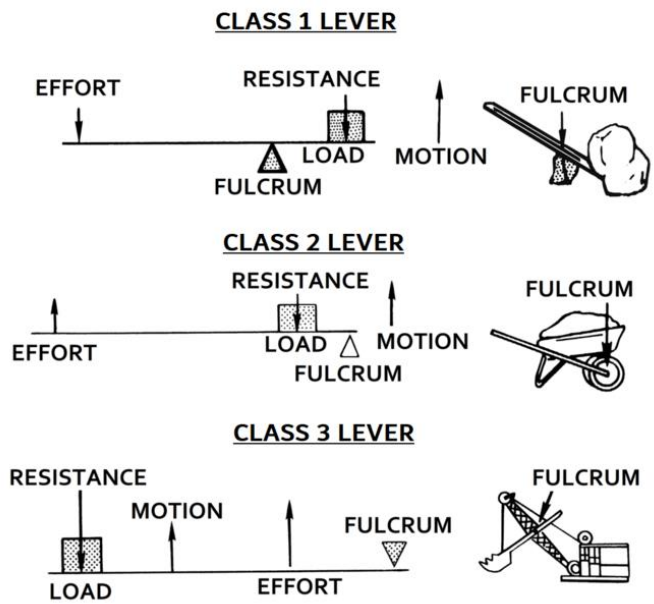
$$F = \frac{420.0 \text{ N}}{6} = 70.0 \text{ N} ,$$

which is 16 pounds of force.

$$\boxed{F = 70.0 \text{ N}}$$

But we do not get something for free. The lady has to apply her force over a greater vertical distance to get the load to move up vertically just a little. Conservation of energy requires the work to be the same. She will push down through 6x the distance the load moves up. In this way, the work is the same and energy is conserved.

The arrangement of the lever in this problem, where the fulcrum is between the applied effort force and the load, is called a Class I Lever. The figure below illustrates the three classes of the lever. For Class II the load is between the effort and fulcrum. The mechanical advantage for the Class I and Class II levers is greater than 1, while for the Class III where the effort is between the load and fulcrum, is less than 1. But there is use for Class III levers.



Courtesy the Archives of Pearson Scott Foresman donated to the Wikimedia Foundation.
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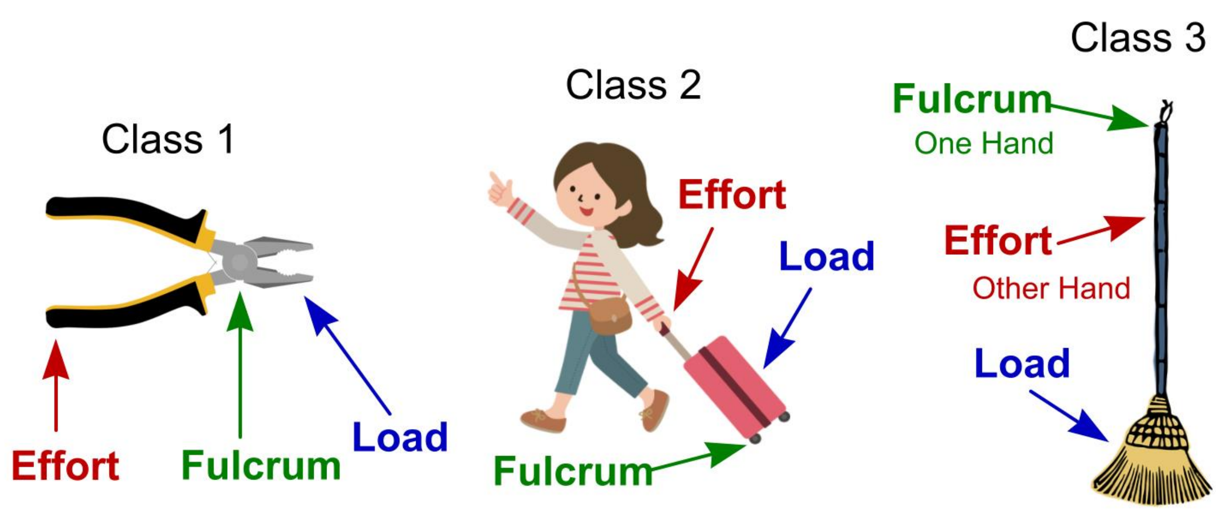
A Class III Lever is good wherever we want the load to move through a greater distance than the applied force. When you lift at the effort point rotating the horizontal arm clockwise, the load moves through a greater distance. In such a case, you are willing to apply a greater force than the load in order to move the load over a larger distance.

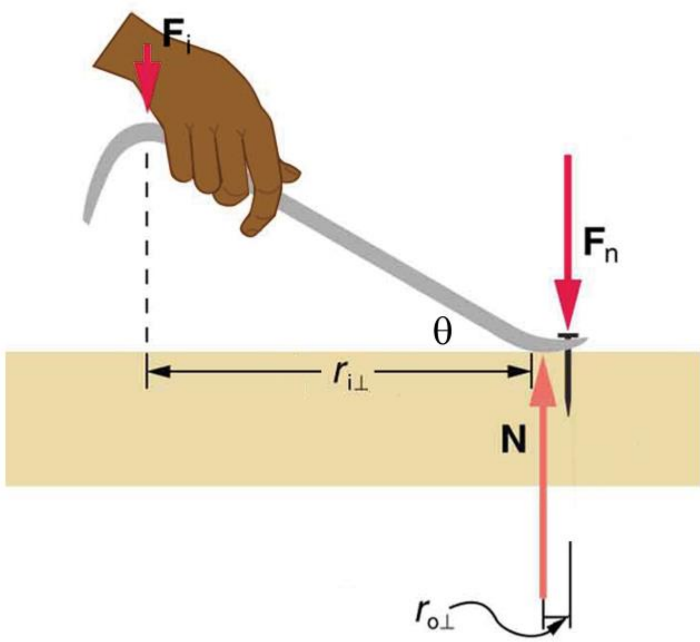


Lever Classes. Clipart Courtesy publicdomainvectors.org

O3. The Nail Puller – Class 1 Lever.

Problem. A nail puller is used in the figure below to remove a nail. An input force F_i is applied on a nail puller at a distance of $r_i = 32.0\ \mathrm{cm}$ along the slanted bar from the pivot. The nail is $r_o = 1.4\ \mathrm{cm}$ to directly to the right of the pivot contact point. The bar makes an angle $\theta = 18°$ with respect to the horizontal. What input force is needed to pull out a nail requiring an output force of $F_o = 710.0\ \mathrm{N}$ (160 lb)? **Hint:** The general torque formula $\vec{\tau} = \vec{r} \times \vec{F}$ has a magnitude of $rF \sin\phi$, where the angle between the two vectors is ϕ. This magnitude can also be expressed as

$$(r \sin\phi)F = r_\perp F \qquad \text{or} \qquad r(F \sin\phi) = rF_\perp .$$



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Solution. They gave us a nice diagram with the forces labeled. We need the forces on the nail puller.

$\vec{F_i}$ - the force due to the hand, which is the input force,

$\vec{N}$ - the normal force the wood pushes up on the nail puller,

$\vec{F_n}$ - the force the nail exerts on the nail puller, which force is equal and opposite to the output force $\vec{F_o}$.

Pick the pivot point to be where the nail puller makes contact with the wood. The equations for equilibrium are given below as a pair of equations: translational and rotational. Take clockwise to be positive for the torques.

$$\begin{Bmatrix} \sum F = N - F_i - F_n = 0 \\ \sum_{\circlearrowright} \tau = F_n r_o - F_i r_i \cos\theta = 0 \end{Bmatrix}$$

But we know from practice that we want the torque equation.

$$F_n r_o = F_i r_i \cos\theta$$

The question wants us to find F_i.

$$\boxed{F_i = \frac{F_n r_o}{r_i \cos\theta}}$$

$$F_i = \frac{F_n r_o}{r_i \cos\theta} = \frac{F_o r_o}{r_i \cos\theta} = \frac{710.0 \text{ N} \cdot 1.4 \text{ cm}}{32.0 \text{ cm} \cdot \cos 18°} = \frac{994.00 \text{ N}}{30.434} = 32.66 \text{ N}$$

$$\boxed{F_i = 32 \text{ N}}$$

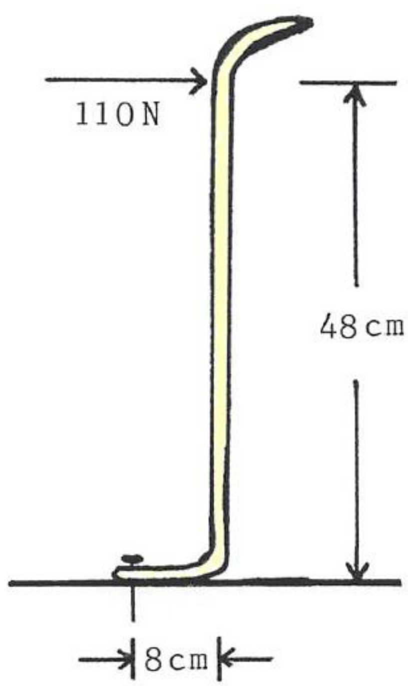
This force is equivalent to 7.2 lb.

The mechanical advantage is

$$\text{M.A.} = \frac{r_{i\perp}}{r_{o\perp}} = \frac{32.0 \text{ cm} \cdot \cos 18°}{1.4 \text{ cm}} = 22.857 \cdot \cos 18° = 21.738$$

$$\boxed{\text{M.A.} = 22}$$

O4. The Crowbar – Class 1 Lever.



A crowbar appears at the left. An input effort force of $F_i = 110.0 \text{ N}$ is applied to the crowbar at a distance of 48 cm above a wooden floor in order to remove a nail. The nail is 8 cm left of where the vertical line of the long part of the crowbar extends to the floor. What is the output force?

Choose the fulcrum pivot at the lower bend of in the crowbar.

$$F_i r_i = F_o r_o$$

$$F_o = F_i \frac{r_i}{r_o} = 110 \text{ N} \cdot \frac{48 \text{ cm}}{8 \text{ cm}} = 110 \cdot 6 \text{ N} \quad \Rightarrow \quad \boxed{F_o = 660 \text{ N}}$$

By the way, the mechanical advantage is $\frac{r_i}{r_o} = \frac{48 \text{ N}}{8 \text{ N}} = 6$.

O5. The Hammer. The hammer can be used as a Class 3 Lever and a Class 1 Lever. We consider each case below.

Class 3 Lever – Hammering Nails.

Courtesy Wikitiki89, Wikimedia License: [Attribution-ShareAlike 3.0 Unported]

When you hammer a nail you are using the hammer as a Class 3 Lever. Your effort is between the fulcrum and the resistance.

Remember for a Class 3 Lever, your mechanical advantage is less the 1. Your extra effort allows for greater motion at the output end, the end where you meet with resistance.

Problem. Estimate the mechanical advantage for the hammer in the figure.

Solution. There is a tendency to panic when precise numbers are not given. First relax and get into estimating mode. The size of my hand when I make a fist is about 10 cm. Therefore, I will take the distance from the fulcrum to the effort vector illustrated in the figure as $r_{\text{effort}} = 10 \text{ cm}$.

But wait. We can answer this directly by estimating the ratio of

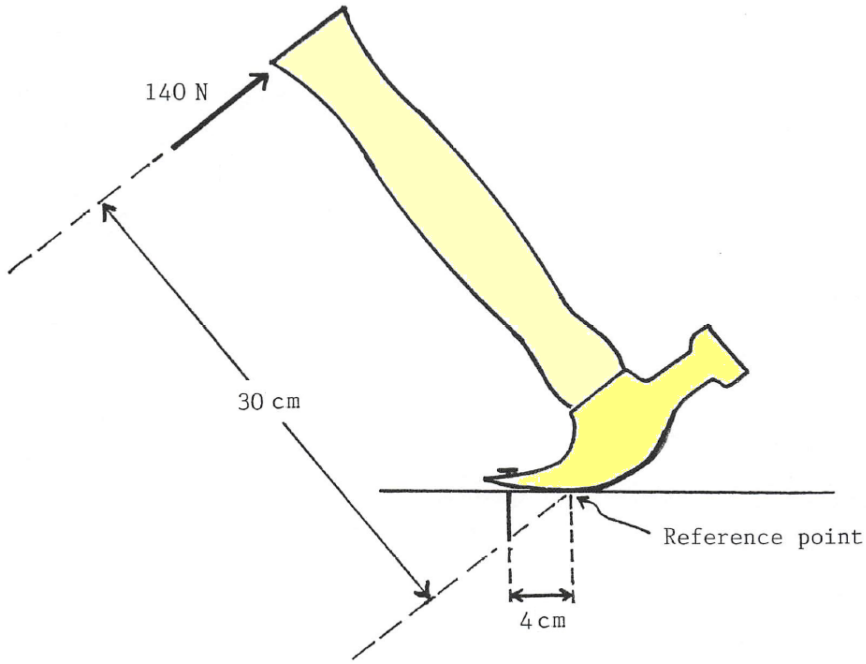
$$\frac{r_{\text{effort}}}{r_{\text{resistance}}} .$$

It is not cheating to place a ruler up to the computer screen to ascertain this ratio if you are uncomfortable in eyeballing the ratio.

My measurement gets me one third.

$$\boxed{\text{M.A.} = \frac{r_{\text{effort}}}{r_{\text{resistance}}} = \frac{1}{3}}$$

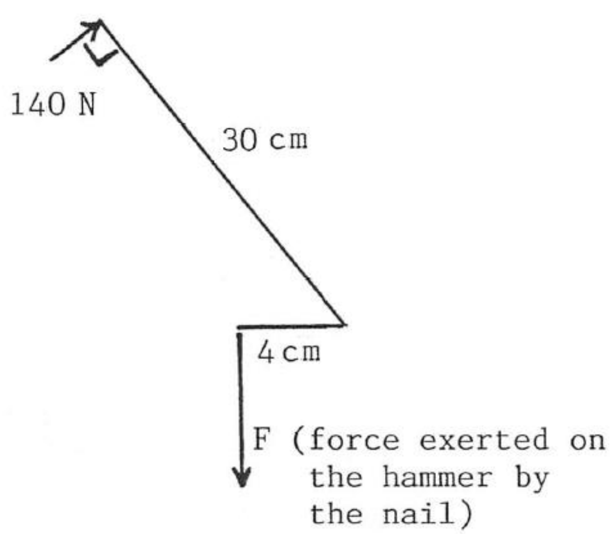
Class 1 Lever – Pulling Nails.



When a hammer is used to pull nails, it is a Class 1 Lever, placing the fulcrum between the input force and the output force.

Problem. Give the mechanical advantage for the hammer pulling a nail at the left and calculate the output force on the nail.

The input force is 140 newtons (31 pounds), acting 30 centimeters (12 inches) from the reference pivot point. The nail is 4 centimeters (1.6 inches) from the reference point.



Solution. Using the reference point in the figure,

$$\text{M.A.} = \frac{r_{\text{effort}}}{r_{\text{resistance}}} = \frac{30 \text{ cm}}{4 \text{ cm}} = \frac{15}{2} = 7.5$$

$$\boxed{\text{M.A.} = 7.5}$$

The output force is

$$F_{\text{output}} = \text{M.A.} \times F_{\text{input}}$$

$$F_{\text{output}} = 7 \times 140 \text{ N}$$

$$\boxed{F_{\text{output}} = 980 \text{ N}}$$

This force is equivalent to 220 pounds.

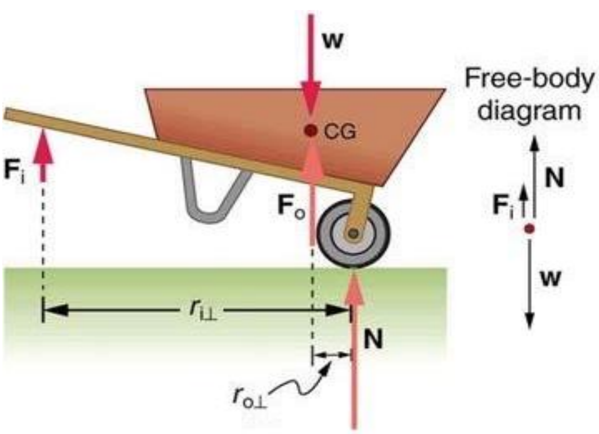
Note that in the diagram F is the force exerted on the hammer by the nail. It is also equal in magnitude to the force exerted by the hammer on the nail, i.e., F_{output}.

O6. The Wheelbarrow. A wheelbarrow has the load between the effort and the fulcrum. Such an arrangement is called a Class II Lever.

Problem 1. Carrying Wood. The wheelbarrow below is loaded with wood.

Courtesy Breaking New Ground, flickr. License: Attribution 2.0 Generic.

See the next figure for a wheelbarrow problem labeled with some parameters.



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The free-body diagram shows the three forces that act on the wheelbarrow: the weight w at the center of gravity, the normal force N at the wheel, and the input effort force F_i.

The output force F_o is the amplified force due to the mechanical advantage. But always focus on the wheelbarrow and the forces acting on it: in this case F_i, w, and N. The way to avoid confusion is to think of F_o at the CG point as replacing F_i.

Problem 1. An input force F_i is applied at a perpendicular distance $r_{i\perp} = 1.25 \text{ m}$ (4.1 ft) from the fulcrum. The center of gravity of the wheelbarrow is $r_{o\perp} = 16.7 \text{ cm}$ (0.5 ft) to the left of the fulcrum. The weight of the wheelbarrow containing the wood is $w = 1250 \text{ N}$ (281 lb).

(i) Find the mechanical advantage.
(ii) Find the input force F_i. For two handles, each hand will supply $F_i / 2$.
(iii) The normal force N.

Solution 1.

(i) Mechanical Advantage. Note that the fulcrum as located where the normal force is located. It is natural to pick the fulcrum as the reference point for the torques. In fact, the diagram with the distances measured from the fulcrum are assuming you will proceed in this fashion since it is the simplest path to the answer.

$$\text{M.A.} = \frac{r_{i\perp}}{r_{o\perp}} = \frac{1.25 \text{ m}}{16.7 \text{ cm}} = \frac{125 \text{ cm}}{16.7 \text{ cm}} = 7.48503$$

$$\boxed{\text{M.A.} = 7.49}$$

(ii) Input force F_i.

$$\sum_{\circlearrowright} \tau = F_i r_{i\perp} - w r_{o\perp} = 0$$

$$F_i r_{i\perp} = w r_{o\perp} \quad \Rightarrow \quad F_i = w \frac{r_{o\perp}}{r_{i\perp}} = 1250 \text{ N} \cdot \frac{1}{\text{M.A.}} = \frac{1250 \text{ N}}{7.48503} = 167.0 \text{ N}$$

$$\boxed{F_i = 167 \text{ N}}$$

This force is equivalent to 37.5 lb.

(iii) The normal force N.

$$\sum F = F_i + N - w = 0$$

$$N = w - F_i$$

$$N = 1250 - 167 = 1083 \text{ newtons}$$

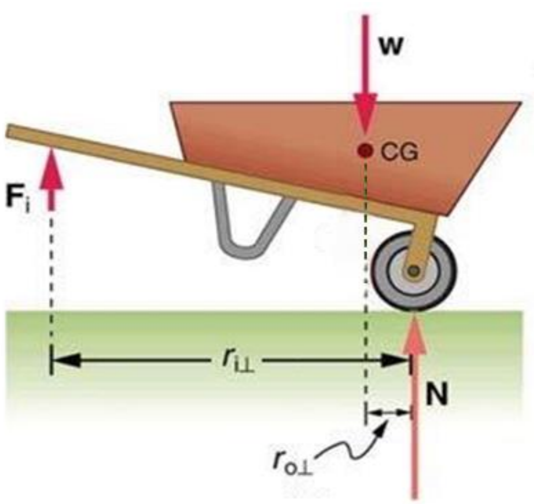
$$\boxed{N = 1080 \text{ newtons}}$$

I wrote out newtons since I though $N = 1083 \text{ N}$ looked a little confusing.

Problem 2. We visit Haiti for our next problem, a country where the wheelbarrow is "integrated deep into Haitian culture, Haitian society and Haitian soul." *Haiti Observer*, June 22, 2013.

A Woman Transporting a Wheelbarrow of Goods in Haiti

Courtesy Alex Proimos, flickr. License: [Attribution-NonCommercial 2.0 Generic]



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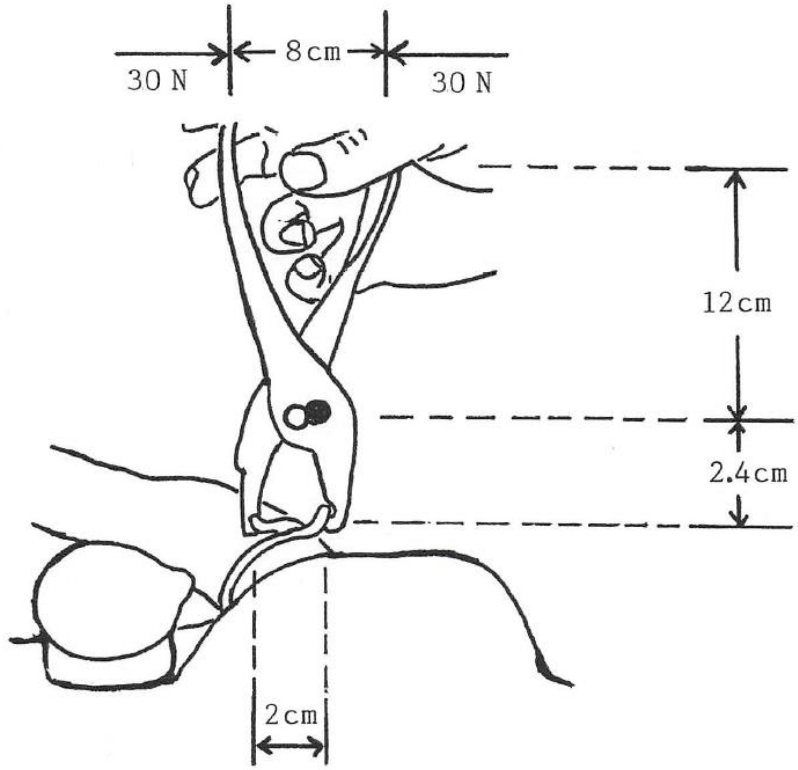
What effort force must the applied by each arm to transport a wheelbarrow full of goods with the following data? The total weight of the goods and wheelbarrow acting at the center of gravity is equal to $w = 360$ N (81 lb). The fulcrum is at the center of the wheel. The applied effort input force has a perpendicular distance $r_{i\perp} = 120$ cm (3.9 ft) from the fulcrum. The center of gravity of the wheelbarrow has a perpendicular distance $r_{i\perp} = 28$ cm from the fulcrum.

Solution 2. The torque equation is $F_i r_{i\perp} = w r_{o\perp}$.

$$F_i = w\frac{r_{o\perp}}{r_{i\perp}} = 360 \cdot \frac{28}{120} = 3 \cdot 28 = 84 \text{ N} \quad \Rightarrow \quad \boxed{\frac{F_i}{2} = 42 \text{ N} = 9.2 \text{ lb}} \quad \text{Each Hand/Arm.}$$

O7. Pliers.



Problem. A mechanic is using a pair of pliers to remove a spring-type clamp from a radiator hose (see figure). If she exerts a force of 30 N at the upper end, what is the force delivered to each end of the clamp?

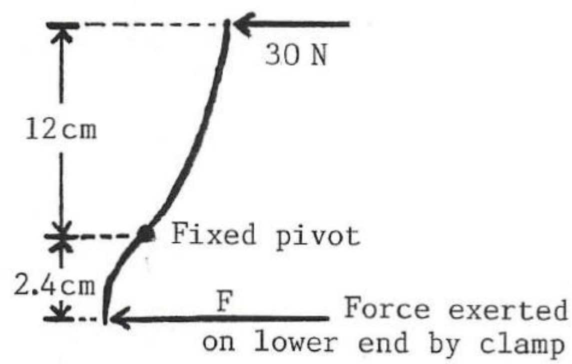
Solution. The torque equation for rotational equilibrium is

$$(30\text{ N})(12\text{ cm}) = F(2.4\text{ cm}) .$$

Solve for the force F .

$$F = \frac{(30\text{ N})(12\text{ cm})}{2.4\text{ cm}}$$

$$\boxed{F = 150\text{ N}}$$



Simplified sketch for half of pliers

See the figure at the left for one half of the pair of pliers. The distances are perpendicular distances, which means we can multiply them by their respective forces to get the torques since force and distance form 90°. In the general torque equation, the magnitude of a torque is

$$Fr\sin\theta ,$$

which we can write as $Fr_{\perp}$,

The mechanical advantage of these particular pliers is

$$\boxed{\text{M.A.} = \frac{12}{2.4} = 5}$$

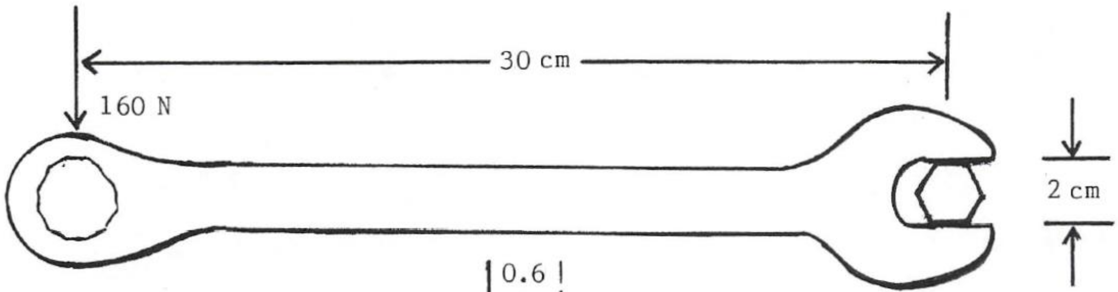
After the mechanic removes the damaged hose, she attaches a new hose with a screw-type clamp, thereby fastening the new hose more securely to the radiator.

O8. Wrench.

Problem 1. Combination Wrench. In this chapter I have been including some equilibrium problems I put together in the 1980s for a High School Science and Engineering Institute my school, the *University of North Carolina at Asheville*, had several summers. I updated the units to the Metric System.

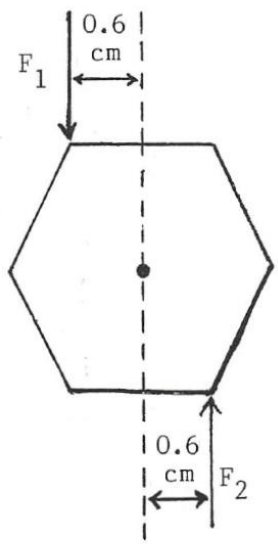
Problem

A mechanic exerts a 160 N force at the end of a wrench in order to remove a hexagonal bolt (see figure). What are the forces F_1 and F_2 if there is



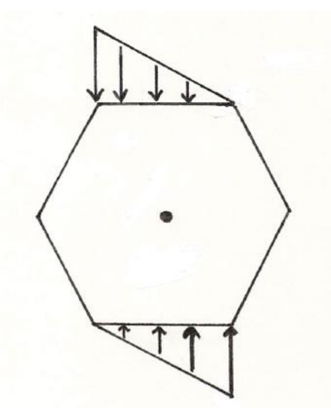
enough clearance so that the only two contact forces are F_1 and F_2?

The wrench and bolt are in equilibrium as the mechanic applies the force. She will need to apply a little more than 160 N in order to remove the bolt.



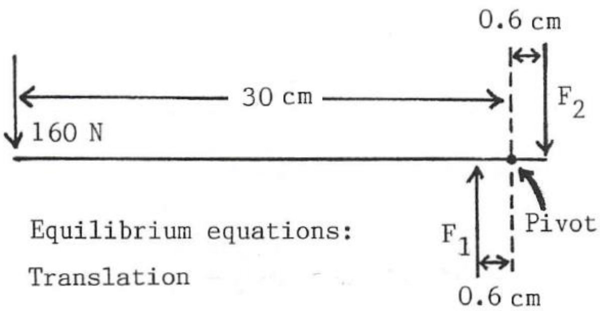
Large view of hexagonal bolt indicating applied contact forces.

Note that the directions of F_1 and F_2 are each reversed when forces on the wrench are desired (see figure below).



In real life, there will be force distributions over the top and bottom of the bolt, where the forces are greater near the corners of the bolt. If we actually had the clearance so the contact forces were only at the extreme edges, most likely the wrench would slip and we would start rounding two corners of the bolt. But the idealization shows the power of physics to calculate subtle parameters. We could made a better model and have a linear force load on each side of the bolt where the forces go from zero at one corner to a maximum at the other. See the figure at the left. That said, let's return to our problem.

Solution 1. For the solution we focus on the wrench. What forces act on the wrench? In the next figure we give a force diagram for the wrench. There are three forces acting on the wrench. Two will have the same magnitudes of the forces acting on the bolt, but in the opposite direction. This feature is due to Newton's Third Law of Action and Reaction.



A force diagram is given at the left for the wrench. Note that forces on the wrench are indicated. The forces that the bolt exerts on the wrench are equal and opposite to the forces that the wrench exerts on the bolt. Therefore, the directions of F_1 and F_2 in the figure at the left (acting on wrench) are opposite to the directions given in the above figure (where forces act on bolt).

Equilibrium equations:

Translation

$$F_1 = 160\text{ N} + F_2$$

Rotation

$$(160\text{ N})(30\text{ cm}) = (F_1)(0.6\text{ cm}) + (F_2)(0.6\text{ cm})$$

We want to solve the two equilibrium equations.

$$\begin{Bmatrix} F_1 = 160\text{ N} + F_2 \\ (160\text{ N})(30\text{ cm}) = F_1(0.6\text{ cm}) + F_2(0.6\text{ cm}) \end{Bmatrix}$$

We can make the equations look simpler by doing a few algebraic steps.

$$\begin{Bmatrix} F_1 - F_2 = 160\text{ N} \\ \dfrac{(160\text{ N})(30\text{ cm})}{0.6\text{ cm}} = F_1 + F_2 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 - F_2 = 160\text{ N} \\ (160\text{ N})(50) = F_1 + F_2 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 - F_2 = 160\text{ N} \\ F_1 + F_2 = 8000\text{ N} \end{Bmatrix}$$

We have two equations with two unknowns. We can easily eliminate F_2 by adding the equations.

$$2F_1 = 160\text{ N} + 8000\text{ N} \quad => \quad 2F_1 = 8160\text{ N} \quad => \quad F_1 = 4180\text{ N}$$

Then we can use either equation from $\left\{ \begin{matrix} F_1 - F_2 = 160 \text{ N} \\ F_1 + F_2 = 8000 \text{ N} \end{matrix} \right\}$ to obtain F_2.

Choosing $F_1 + F_2 = 8000 \text{ N}$ leads to $F_2 = 8000 \text{ N} - F_1 = 8000 \text{ N} - 4180 \text{ N} = 3820 \text{ N}$

$$\boxed{F_1 = \; 4180 \text{ N} = 940 \text{ lb}} \qquad \boxed{F_2 = \; 3820 \text{ N} = 860 \text{ lb}}$$

Note that F_1 is greater than F_2 by the amount of the applied force: $F_1 = 160 \text{ N} + F_2$.

Problem 2. Lug Nut Wrench. The photo illustrates removing a lug nut in the old school fashion, using a four-way lug wrench. I would have to hold that lug nut wrench more towards the ends in order to achieve greater torque.

Courtesy Ivan Radic, flickr
Photo Taken May 23, 2020
License: [Attribution 2.0 Generic]

My Toyota Camery has a lug bolt with a diameter of 21 mm. The Toyota spec on tightening though lists a torque of 76 ft · lb, using the British System of units.

Problem 2. What force F must be applied by each hand in the photo if each hand is 12 cm from the axis and a 76 ft · lb torque is required to loosen the nut? The left hand is pushing down and the right hand is pulling up with an equal magnitude of force.

Solution 2. The total torque exerted by hands is

$$\tau = \tau_{\text{left hand}} + \tau_{\text{right hand}} .$$

$$\tau = F(12 \text{ cm}) + F(12 \text{ cm}) = 76 \text{ ft} \cdot \text{lb}$$

$$F(16 \text{ cm}) = 76 \text{ ft} \cdot \text{lb} \quad => \quad F = \frac{76 \text{ ft} \cdot \text{lb}}{24 \text{ cm}}$$

$$F = \frac{76 \text{ ft} \cdot \text{lb}}{24 \text{ cm}} \frac{30.48 \text{ cm}}{1 \text{ ft}} = \frac{76 \cdot 30.48}{24} \text{lb} = 97 \text{ lb} \quad => \quad F = 97 \text{ lb} \frac{4.448 \text{ N}}{\text{lb}} = 430 \text{ N}$$

I am not strong enough. So I grab the wrench at the far ends, increasing my distances to 17 cm.

The force needed will be lowered. We can adjust things using the factor 12/17.

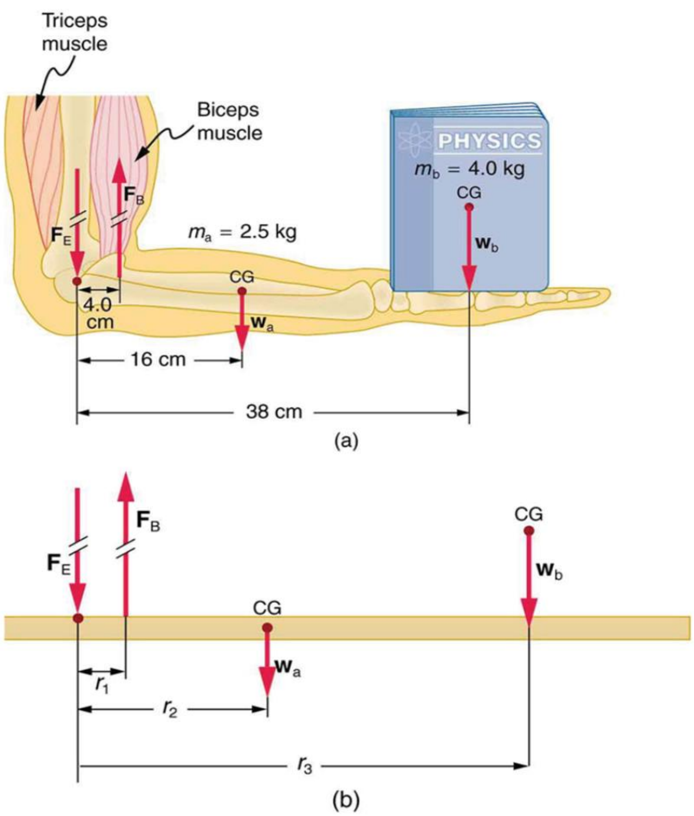
$$F = 97\text{ lb} \rightarrow \frac{12}{17}\cdot 97\text{ lb} = 68\text{ lb}$$

The reduced force is now about 300 newtons for each hand.

Or I might opt to use a power tool. And a torque wrench can be handy – telling you the torque.

O9. Biological Forces.

Problem 1. Arm. A person is holding a mass as shown in the figure with values of the relevant parameters indicated in (a). See the simplified model of the arm as a board shown in (b). What is the force F_B that the bicep muscles exerts on the arm and what is the force F_E that the humerus bone exerts downward on the elbow.



Courtesy OpenStax, Rice University
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Choose the reference for the torques at where F_E acts since then the torque due to F_E will be zero and we will have a torque equation will one less unknown.

$$\begin{Bmatrix} F_B = F_E + w_a + w_b \\ F_B r_1 = w_a r_2 + w_b r_3 \end{Bmatrix}$$

The second equation gives the biceps muscle force.

$$F_B = \frac{w_a r_2 + w_b r_3}{r_1}$$

$$F_B = \frac{m_a g r_2 + m_b g r_3}{r_1}$$

$$F_B = \frac{m_a r_2 + m_b r_3}{r_1} g \quad => \quad F_B = \frac{(2.5\text{ kg})(0.16\text{ m}) + (4.0\text{ kg})(0.38\text{ m})}{0.04\text{ m}}(9.8\frac{\text{m}}{\text{s}^2})$$

$$F_B = \frac{(2.5\text{ kg})(0.16\text{ m}) + (4.0\text{ kg})(0.38\text{ m})}{0.04\text{ m}}(9.8\frac{\text{m}}{\text{s}^2}) = \frac{1.92}{0.04}\cdot 9.8\text{ kg}\frac{\text{m}}{\text{s}^2} = 470.4\text{ N}$$

The force of the humerus on the elbow is found from the first equation $F_B = F_E + w_a + w_b$.

$$F_E = F_B - w_a - w_b \quad \Rightarrow \quad F_E = F_B - m_a g - m_b g \quad \Rightarrow \quad F_E = F_B - (m_a + m_b) g$$

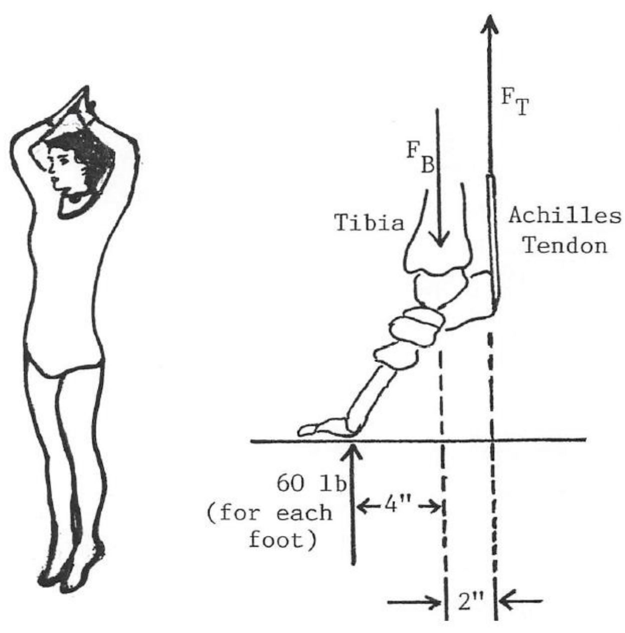
$$F_E = 470.4 \text{ N} - (2.5 \text{ kg} + 4.0 \text{ kg}) 9.8 \frac{\text{m}}{\text{s}^2} \quad \Rightarrow \quad F_E = 470.4 \text{ N} - 63.7 \text{ N} = 406.7 \text{ N}$$

$$\boxed{F_B = 470 \text{ N} = 106 \text{ lb}} \qquad \boxed{F_E = 407 \text{ N} = 91 \text{ lb}}$$

These biological forces are large compared to the weight of the arm and book.

$$w + w_b = (2.5 \text{ kg} + 4.0 \text{ kg}) 9.8 \frac{\text{m}}{\text{s}^2} = 63.7 \text{ N} = 14 \text{ lb}$$

Problem 2. Foot. A 54-kg lady (120 lb) is standing tiptoe. Each foot supports half her weight. The distance from the point of contact on the ground to the vertical line through when the tibia bone pressses downward is 4 inches = 10 centimeters. The corresponding distance to the extended vertical tendon line is 2 inches = 5 centimeters. Find the forces that the tibia F_B and Achilles tendon F_T exert on the foot.



Solution 2. I will work in the British System of Units as I can easily read values off the figure and I will then switch to Metric at the end. Choose the pivot reference to be 4" to the right of where the foot touches the ground. In this way, I can eliminate F_B from the torque equation.

$$\begin{Bmatrix} 60 \text{ lb} + F_T = F_B \\ 60 \text{ lb} \cdot 4" = F_T \cdot 2" \end{Bmatrix}$$

The second equation readily gives

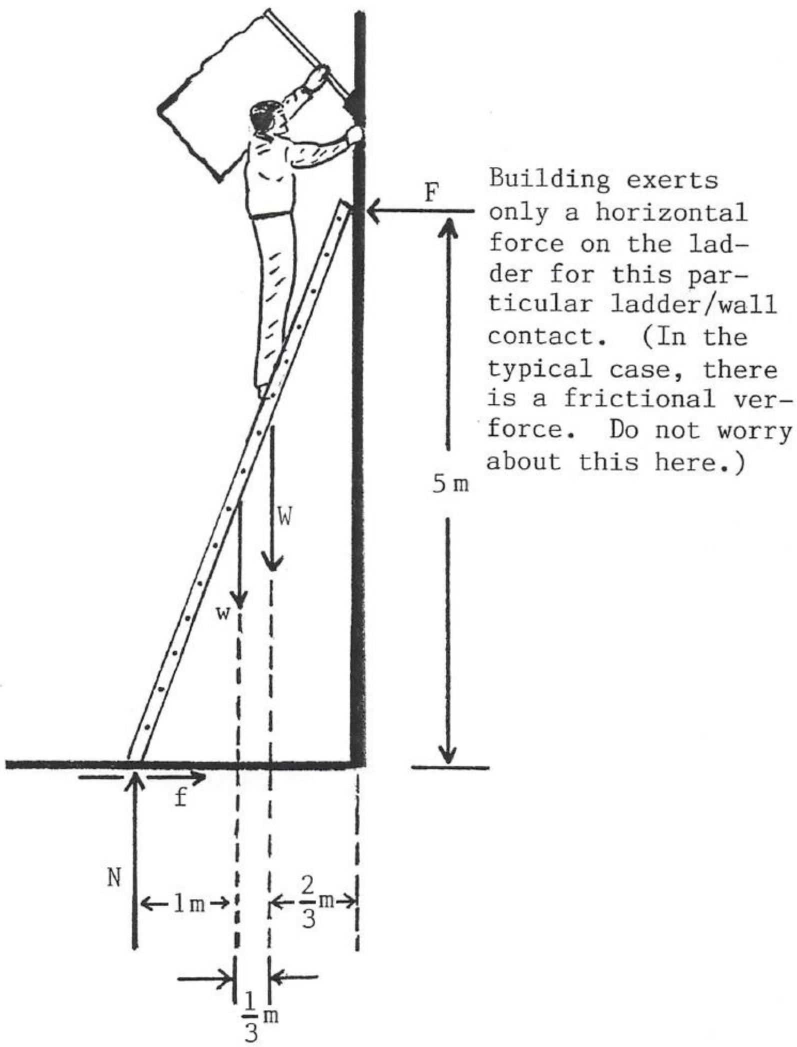
$$F_T = \frac{60 \text{ lb} \cdot 4"}{2"} = 120 \text{ lb}.$$

Then, the first equation gets us $F_B = 60 \text{ lb} + F_T = 180 \text{ lb}$. $\boxed{F_T = 120 \text{ lb}}$ $\boxed{F_B = 180 \text{ lb}}$

To get newtons, multiply each by 4.448. $\boxed{F_T = 530 \text{ N}}$ $\boxed{F_B = 800 \text{ N}}$

If the lady stands on one foot, these values get doubled!

O10. Ladder. A man is standing on a ladder in order to attach a flag to a building. The man has a weight of W = 72.0 kilogram · force (160 lb) and the ladder weighs w = 12.0 kilogram · force (26 lb). The weight of the ladder acts at the center of gravity as indicated in the figure. The force of the building on the ladder is mostly horizontal so you can neglect any vertical forces there. Find the coefficient of static friction μ_s needed between the ladder and the ground to prevent the ladder from sliding down. Also find the normal force N that the ground exerts on the ladder.



It is wise to pick the point of contact with the ground for the reference. In this way, the two unknown forces at the reference do not contribute in the torque equation since they are at the pivot point. The equation for rotational equilibrium gives an equation with F. The two translational equilibrium equations gives equations for f and N.

Let's start with the rotational equilibrium equation to first obtain F.

$$w \cdot 1 + W \cdot \frac{4}{3} = F \cdot 5$$

$$12 \text{ kg}_F \cdot 1 + 72 \text{ kg}_F \cdot \frac{4}{3} = F \cdot 5$$

The unit kg_F stands for kilogram · force.

$$12 \text{ kg}_F \cdot 1 + 72 \text{ kg}_F \cdot \frac{4}{3} = F \cdot 5 \quad => \quad 12 \text{ kg}_F + 96 \text{ kg}_F = 5F \quad => \quad 108 \text{ kg}_F = 5F$$

$$F = \frac{108 \text{ kg}_F}{5} = 21.6 \text{ kg}_F = 21.6 \text{ kg} \cdot 9.8 \ \frac{\text{N}}{\text{kg}} = 211.68 \text{ N}$$

$$\boxed{F = 21.6 \text{ kg}_F = 212 \text{ N} = 48 \text{ lb}}$$

Next we consider translational equilibrium.

Horizontal: $f = F$ => $f = 21.6 \text{ kg}_F$

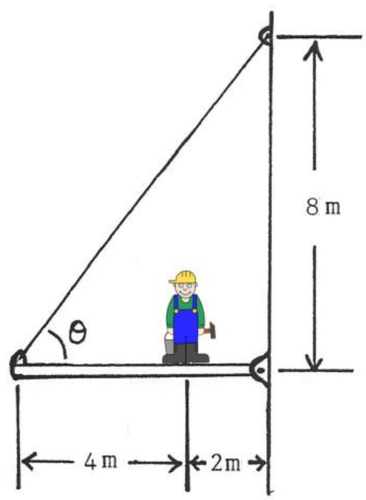
Vertical: $N = w + W$ => $N = 12.0 \text{ kg}_F + 72.0 \text{ kg}_F = 84.0 \text{ kg}_F$

We can now calculate the minimum coefficient of static friction needed.

$$f \le \mu_s N \quad => \quad 21.6 \text{ kg}_F \le \mu_s 84.0 \text{ kg}_F \quad => \quad \frac{21.6 \text{ kg}_F}{84.0 \text{ kg}_F} \le \mu_s$$

$$0.257 \le \mu_s \quad => \quad \boxed{\mu_s \ge 0.26}$$

The minimum coefficient of static friction needed is 0.26.

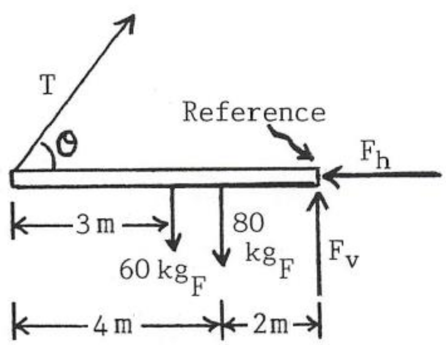


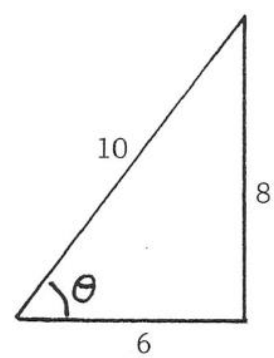
O11. Scaffold. A man is standing on a scaffold ready to work on the exterior of a building. Find the supporting forces on the board if the man weighs 80.0 kilograms · force (about 180 lb). The board weighs 60.0 kilograms · force (about 135 lb).

Man on Scaffold.
Credit Line: Man Clipart publicdomainvectors.org

We start with a force diagram for the floor board.

We pick the reference point to be the place with the most ignorance. You can pick any reference since if there is no rotation, the sum of the torques about any point will be zero.





Reference Triangle

$$\begin{cases} \sum F_x = 0 \Rightarrow T\cos\theta = F_h \\ \sum F_y = 0 \Rightarrow T\sin\theta + F_v = 60 \text{ kg}_F + 80 \text{ kg}_F \\ \sum \tau = 0 \Rightarrow (T\sin\theta)(6 \text{ m}) = (60 \text{ kg}_F)(3 \text{ m}) + (80 \text{ kg}_F)(2 \text{ m}) \end{cases}$$

Note that the weight of the board can be placed at the center of gravity, which is the center of the board. This center is 3 m from each end of the 6-meter board.

We start with the torque equation and use the reference triangle for the sine.

$$(T\sin\theta)(6\text{ m}) = (60\text{ kg}_F)(3\text{ m}) + (80\text{ kg}_F)(2\text{ m})$$

$$(T\frac{8}{10})(6\text{ m}) = 180\text{ kg}_F\cdot\text{m} + 160\text{ kg}_F\cdot 2\text{ m}$$

$$(T\frac{8}{10})(6) = 180\text{ kg}_F + 160\text{ kg}_F\cdot 2$$

$$(T\frac{48}{10}) = 180\text{ kg}_F + 320\text{ kg}_F$$

$$\frac{24}{5}T = 500\text{ kg}_F$$

$$T = \frac{5}{24}500\text{ kg}_F \quad => \quad T = 104.167\text{ kg}_F$$

$$\boxed{T = 104\text{ kg}_F}$$

The horizontal equilibrium equation will get us F_h.

$$F_h = T\cos\theta = (104.167)(\frac{3}{5})\text{kg}_F = 62.5\text{ kg}_F$$

$$\boxed{F_h = 62.5\text{ kg}_F}$$

The vertical equation leads us to F_v.

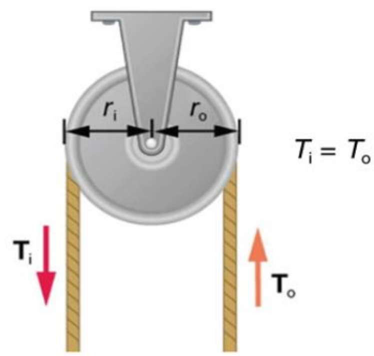
$$T\sin\theta + F_v = 60\text{ kg}_F + 80\text{ kg}_F \quad => \quad F_v = 60\text{ kg}_F + 80\text{ kg}_F - T\sin\theta$$

$$F_v = 140\text{ kg}_F - T\sin\theta \quad => \quad F_v = 140\text{ kg}_F - 104.167(\frac{4}{5})\text{kg}_F = 140\text{ kg}_F - 83.333\text{ kg}_F$$

$$\boxed{F_v = 56.7\text{ kg}_F}$$

To get a pounds estimate, just double the kg_F.

O11. Pulley Systems.



Images are Courtesy OpenStax, Rice University
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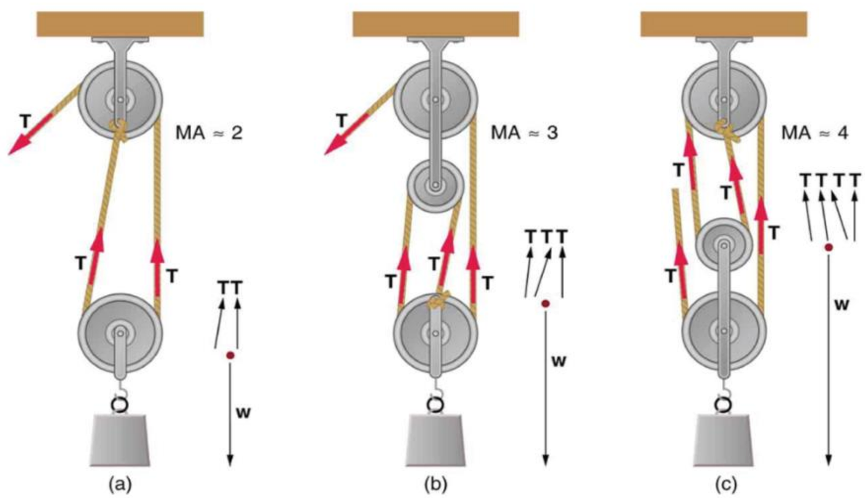
We have often encountered the single pulley in our course. Let "i" stand for input and "o" for output. We consider pulleys that either massless or where the rope slides over the pulley with no friction. The massless pulley is justified when hanging weights are much greater than the pulley mass.

For the single pulley here, the mechanical advantage is 1.

$$\text{M.A.} = \frac{T_o}{T_i} = 1$$



We have also seen in our course a double pulley where we looped the rope. See (a) in the figure at the left. The result is that we can lift twice as much weight. The mechanical advantage is 2.

$$\text{M.A.} = \frac{T_o}{T_i} = \frac{2T}{T} = 2$$

For tripling up on the rope using three pulleys, the mechanical advantage is 3. See (b).

$$\text{M.A.} = \frac{T_o}{T_i} = \frac{3T}{T} = 3$$

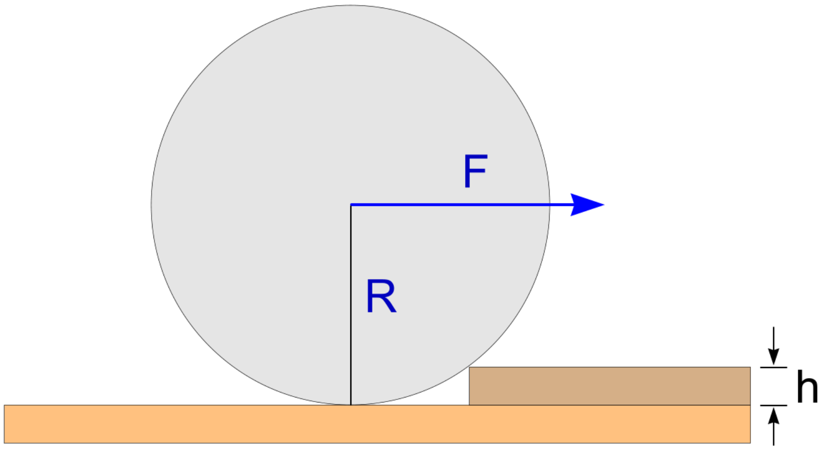
With the clever attachment in (c) with 3 pulleys, we can get a mechanical advantage of 4.

$$\text{M.A.} = \frac{T_o}{T_i} = \frac{4T}{T} = 4$$

You can imagine the far left rope in (c) being looped over an additional pulley to the left. You still get the same mechanical advantage. A fast rule to get the mechanical advantage in all cases is to count how many ropes are pulling upward.

O12. The Wheel and the Step.

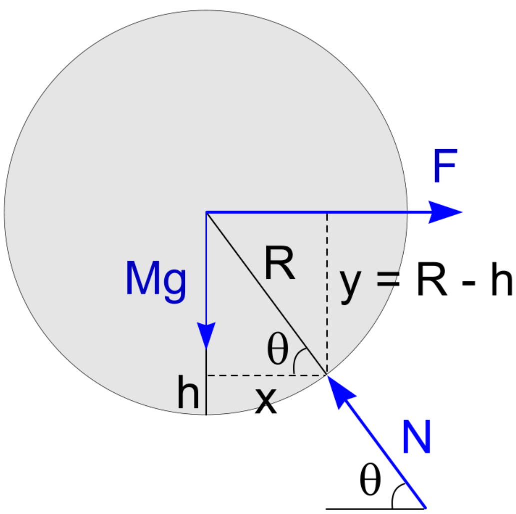
Problem. What is the minimum force F needed to get the wheel to begin to climb up the step of height h?



We start with a force diagram. There are three forces acting on the wheel when the wheel just leaves the ground.

1. The applied force F.
2. Gravity Mg.
3. Slanted normal force N.

See the force diagram below.



Solution Method 1. We use the translational equilibrium equations.

$$\begin{Bmatrix} \sum F_x = F - N\cos\theta = 0 \\ \sum F_y = N\sin\theta - Mg = 0 \end{Bmatrix}$$

$$\begin{Bmatrix} F = N\cos\theta \\ N\sin\theta = Mg \end{Bmatrix}$$

$$\begin{Bmatrix} N\cos\theta = F \\ N\sin\theta = Mg \end{Bmatrix}$$

Divide $N\sin\theta = Mg$ by $N\cos\theta = F$.

$$\frac{N\sin\theta}{N\cos\theta} = \frac{Mg}{F} \quad \Rightarrow \quad \frac{\sin\theta}{\cos\theta} = \frac{Mg}{F} \quad \Rightarrow \quad \tan\theta = \frac{Mg}{F} \quad \Rightarrow \quad F = \frac{Mg}{\tan\theta}$$

$$\tan\theta = \frac{y}{x} = \frac{y}{\sqrt{R^2 - y^2}} = \frac{R-h}{\sqrt{R^2 - (R-h)^2}}$$

$$\tan\theta = \frac{R-h}{\sqrt{R^2 - (R^2 - 2Rh + h^2)}} \quad \Rightarrow \quad \tan\theta = \frac{R-h}{\sqrt{R^2 - R^2 + 2Rh - h^2}}$$

$$\tan\theta = \frac{R-h}{\sqrt{2Rh-h^2}}$$

$$F = \frac{Mg}{\tan\theta} \quad \Rightarrow \quad F = \frac{Mg}{(R-h)/\sqrt{2Rh-h^2}} \quad \Rightarrow \quad F = \frac{Mg\sqrt{2Rh-h^2}}{R-h}$$

$$\boxed{F = \frac{Mg\sqrt{2Rh-h^2}}{R-h}}$$

Solution Method 2. We use the rotational equilibrium equation. Pick the normal contact point as the reference and clockwise as positive. Using $r_\perp$ distances for the torques,

$$\sum_{\circlearrowright} Fy - Mgx = 0 \quad \Rightarrow \quad Fy = Mgx \quad \Rightarrow \quad F = Mg\frac{x}{y}.$$

$$F = Mg\frac{x}{y} = Mg\cot\theta$$

Since $\tan\theta = \frac{y}{x} = \frac{y}{\sqrt{R^2-y^2}} = \frac{R-h}{\sqrt{R^2-(R-h)^2}} = \frac{R-h}{\sqrt{R^2-(R^2-2Rh+h^2)}} = \frac{R-h}{\sqrt{R^2-R^2+2Rh-h^2}}$

and $\tan\theta = \frac{R-h}{\sqrt{2Rh-h^2}}$ we find the same answer as before: $F = Mg\cot\theta = \frac{Mg\sqrt{2Rh-h^2}}{R-h}$.

Is the answer reasonable?

The dimensions check out

$$[F] = [Mg]\frac{[\sqrt{2Rh-h^2}]}{[R-h]} = [Mg]\frac{L}{L} = [Mg]$$

What about if the step height approaches zero?

$$\lim_{h\to 0} F = \lim_{h\to 0}\frac{Mg\sqrt{2Rh-h^2}}{R-h} = Mg\frac{\sqrt{2R\cdot 0-0^2}}{R-0} = Mg\cdot\frac{0}{R} = 0$$

This result is expected. It gets easier and easier as the step height gets smaller and smaller.