F. Frequency Ratios

We study in this chapter the fundamental topic of frequency ratios. A frequency ratio is relevant when we consider two tones. Some combinations are pleasant, others are unpleasant. The absolute frequency is not important. It is the relative frequency that is significant. You know this. For example, you can sing a song like *Mary Had a Little Lamb*. Then you can start the song again at a slightly higher pitch. You will unconsciously adjust all the subsequent pitches to their proper relative positions so that the song once again sounds like *Mary Had a Little Lamb*. You have preserved the song. This is called *transposing*.

Pianists are often asked to play the same song starting at different pitches in order to accommodate singers. The pianist is said to *transpose* the song into another key, i.e., another starting point. The word transpose is used because the song is often written in a different key. The musician is not reading the music, not playing where it is written, but transposing to a different key. This takes practice. Of course, if the pianist can play by ear, music is not necessary in the first place.

Over the ages people of different cultures have chosen tones that coordinate well with each other. These groups of tones are called *scales*. We will work with the *western major musical scale* in this chapter. This is the scale you learned many years ago: *Do-Re-Mi-Fa-Sol-La-Ti-Do'*. It's all the music you need to know in order to understand this text. You probably learned it in kindergarten. You see, kindergarten is very important. You learn about scales, among other important activities like relating to others. Someone once said that you learn in kindergarten everything you need to succeed in life.

The musical scale can start on any pitch. You sing *Do* anywhere you like and then proceed to sing the other tones accordingly. Once again, the relative pitch is important for the study of the coordination of these tones. Therefore, we will compare the frequencies by ratios. Ratios are not complicated. Let's use a monetary analogy. You may have $150 and your friend has $300. Well, we only care that your friend has twice as much as you if we are only interested in the ratio. We can say that you have a given amount, which we describe by 1. The "one" simply means the money in your pocket, your "one" pocket of money. Then we say your friend has 2, meaning the equivalent of 2 times what you have. Your friend has "two pockets" of money in a sense. Another person might have 3 or 4 times what you have. We are comparing amounts of money with what you have. The numbers are therefore ratios.

We will proceed first by reviewing the musical scale. We will develop the concept of *musical intervals*. We will develop a trick where we can readily tell which note in the scale is played relative to our reference note *Do*. Secondly, we will develop a technique for measuring frequency ratios. This is the subject of *Lissajous* (LISS-uh-joo) *figures*. Thirdly, we will apply this technique to determine the frequency ratios for the tones in our 8-note musical scale. But we will work backwards at that point. We will first search for tones with simple ratios. Then we will discover that these notes are in our scale. The scale we unfold will be based on perfect ratios. In a later chapter we will learn that, today, our instruments are not tuned perfectly to such a scale. Compromises are made.

Musical Intervals

Let's review a couple of definitions before we investigate musical intervals. Frequency is a measure of how rapidly vibrations occur. The frequency of a sound wave is perceived as pitch. A *musical scale* is built from a set of frequencies called *tones* or *notes*.

Frequency (pitch) - number of vibrations per second (hertz, Hz).

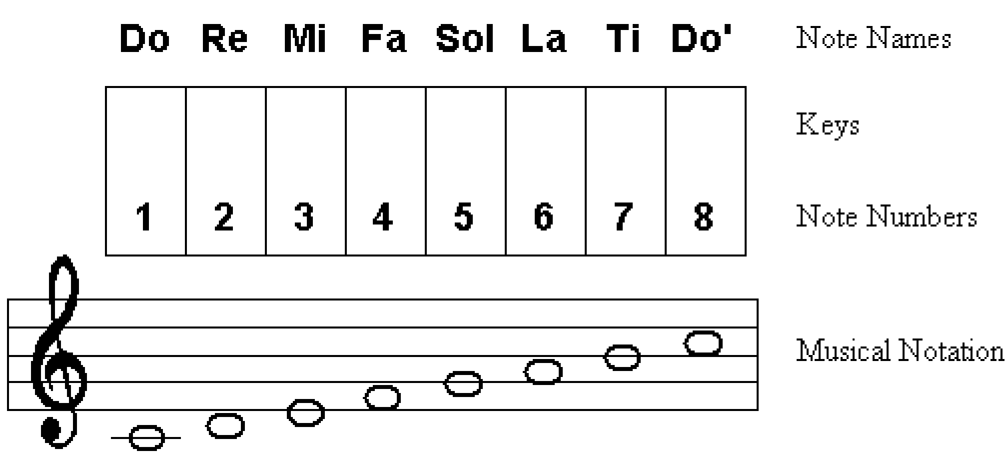
Musical scale - a discrete set of frequencies (tones or notes).

The common major scale is illustrated below in Fig. F-1. There are eight tones in the scale. These are numbered in the figure. It will be convenient for us to refer to a tone by its number from time to time. The first note is *Do*, the second note is *Re*, and so on. We call the last note *Do'* to distinguish it from the first note, *Do*.

You will find musical notation below the keys in Fig. F-1. You do not need to worry if this is the first time you have seen such a thing. Consider it like a thermometer that indicates pitch without using numbers. This "musical thermometer" is called a *staff*. The higher up on the staff, the higher the pitch of the note. In other words, the higher the frequency, the higher the position. But don't think about numbers. The notes, of course, have numbers to define frequencies, but the spacing on the musical staff does not correspond to them in some simple fashion. Remember the AM numbers on our radio. They were spaced unevenly. Such a spacing where equal steps in distance do not correspond to equal steps in frequency is called a *nonlinear scale*.

The beauty of the nonlinear musical staff is that each note of our scale is equally spaced by position, not frequency. The tones alternately fall on lines or spaces. Notice the short line necessary to indicate the first note. The symbol at the left of the staff is called a *clef* symbol. Ours is a *treble clef,* meaning tones in the upper half of the piano.

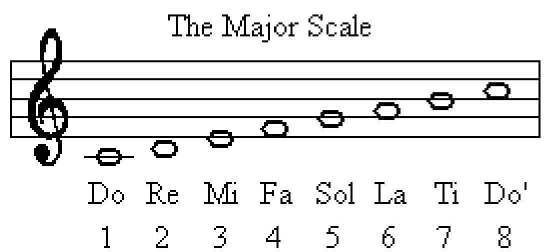
Fig. F-1. The Major Scale.



We take the first note as our reference tone. We will compare the other notes to *Do*. For example we may compare *Sol* to *Do*. Moving from the reference *Do* up to *Sol*, we move up to the 5th note. Musicians say we move up a 5th. If you press these two keys together, you are playing an interval called the 5th. If you choose to play the 4th note (*Fa*) with the reference (*Do*), you are playing a 4th. Going from *Do* to *Fa* defines an interval of a 4th. So an interval is really the "musical distance" between two notes. We will always start on the *Do*.

Since we jump from *Do* to another note, it can be difficult to recognize the upper note if someone plays one for us and asks us to identify it. Music teachers have come up with a technique to help us do this fairly accurately with a little practice. We use the old trick of remembering the start of a song for each of the jumps. Then, when we hear a jump, we scan through the songs in our head until we find a match. The actual practice of doing this is called *ear training*. Table F-1 below lists the 8 intervals and a song to help us recognize each of them.

Table F-1. Intervals.



Begin-End	Interval	Reference Theme
Do-Do	Unison	"America"
Do-Re	Second	"Doe a Deer"
Do-Mi	Third	"Marine's Hymn"
Do-Fa	Fourth	"Here Comes the Bride"
Do-Sol	Fifth	"Twinkle, Twinkle, Little Star"
Do-La	Sixth	"My Bonnie"
Do-Ti	Seventh	"Superman Movie Theme"
Do-Do'	Octave	"Somewhere Over the Rainbow"

As an example, consider the interval of a fourth. This corresponds to playing the first note *Do* as always and then the 4th note *Fa*. These notes come at the beginning of the bridal march by Wagner, played at the start of weddings. These notes may be played simultaneously to perceive how well they blend. Are they pleasing? We will answer such questions in section F-3. The 7th in "Superman" does not come at the very beginning of the theme.

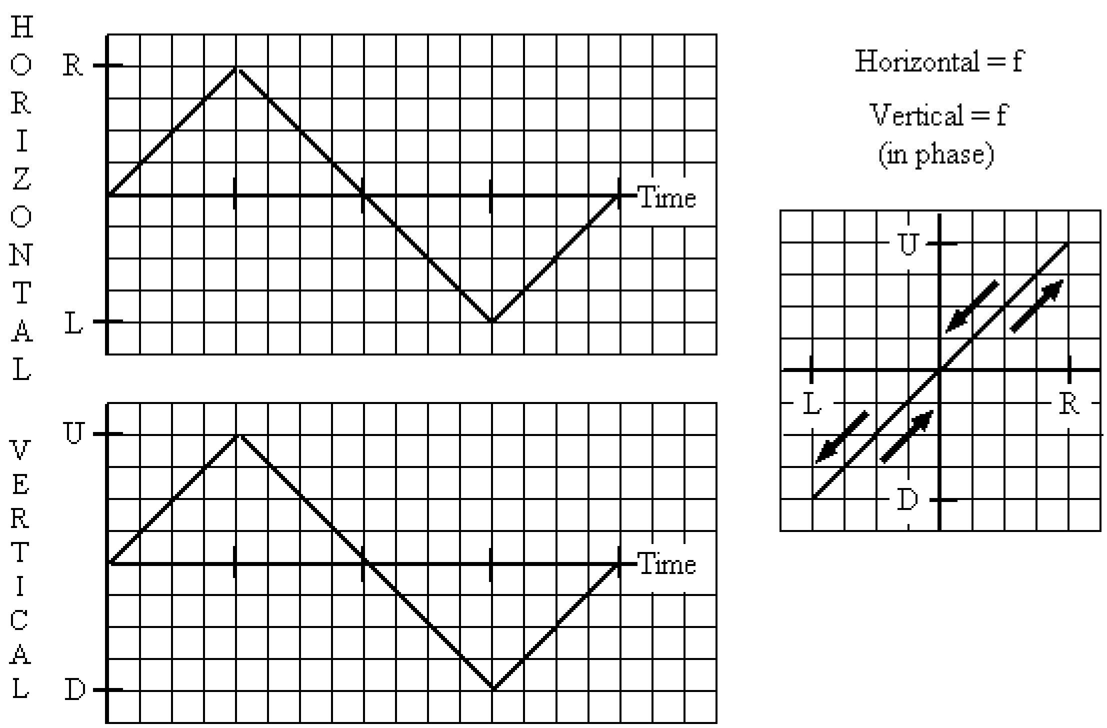
Lissajous Figures

Lissajous (LISS-uh-joo) figures are part of the language of the scientist, i.e., physicist or engineer. We have just discussed components of the language of the musician. In this section we turn to physics. Then in the third section we combine both the musician's and physicist's techniques in a wonderful experiment to determine the frequency ratios of the perfect major scale.

Our analysis will have a blend of mathematics, physics, perception, music, esthetics, and even philosophy. The Greek mathematician and mystic, Pythagoras, was one of the first to study the esthetics of pleasing tone combinations. He showed that tones with the simplest mathematical ratios were the most pleasing. This had profound philosophical implications for him. He discovered that one way to understand the beauty and harmony of nature is through mathematics. The methods we develop here will enable us to pursue the study of harmonious combinations of tones.

We will explain Lissajous figures by example. Consider a party game where you are given two sets of instructions for walking on a floor, the playing field. One set of instructions tells you how to move East or West, which we call right-or-left, while the other tells you how to move North or South, which we simply call up-or-down. You must move simultaneously according to both instructions. Fig. F-2a below shows our first game. The instructions are at the left in graphical form, while the playing field is pictured at the right. R stands for Right, L for Left, U for Up, and D for Down.

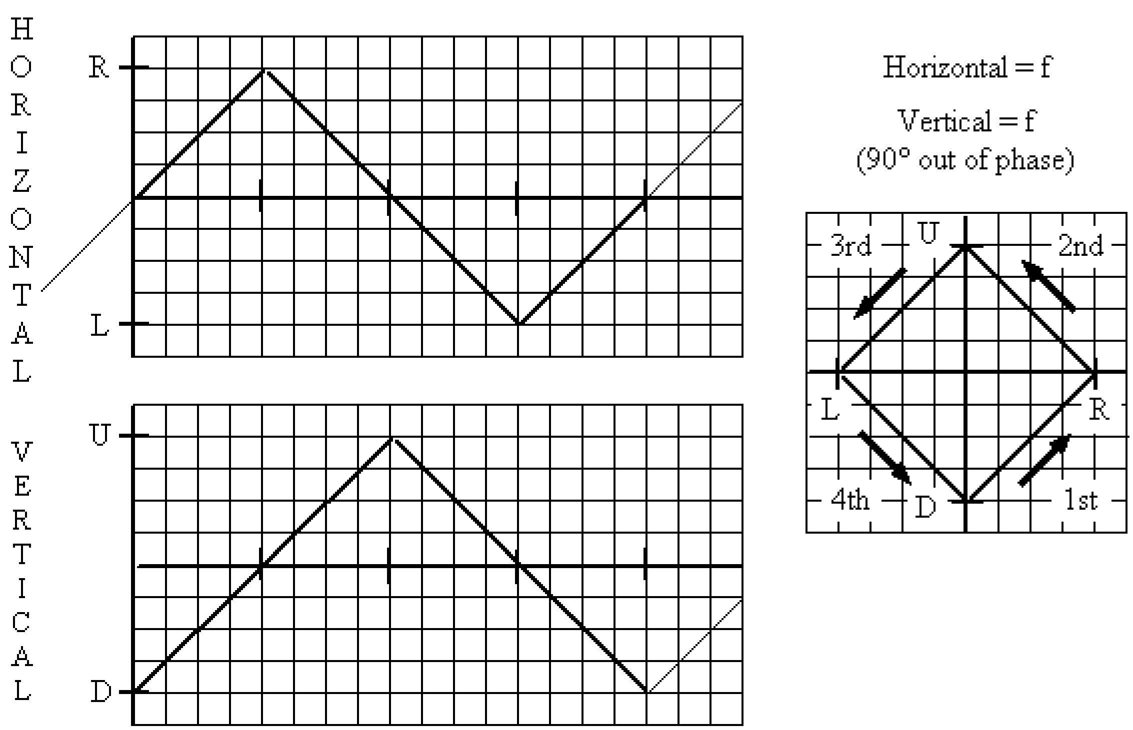
Fig. F-2a. Lissajous Figures: Case I.



The game proceeds in four phases or quarters. The horizontal instructions tell you how much to move left or right, while the vertical instructions tell you how to move up or down. For the 1st quarter, your horizontal instructions tell you to start in a position neither right nor left, then proceed 4 paces (blocks) to the right. Now you must do this while at the same time following the vertical instructions, which tell you to move 4 paces up (from the center) during the 1st quarter. To perform both of these actions simultaneously, you walk along a diagonal to a destination that is to the right and up. The 2nd quarter calls for you to come back to the center position, a position neither right nor left, neither up nor down. The 3rd quarter has you going to the left and down at the same time. The 4th quarter instructs you to come back to the center.

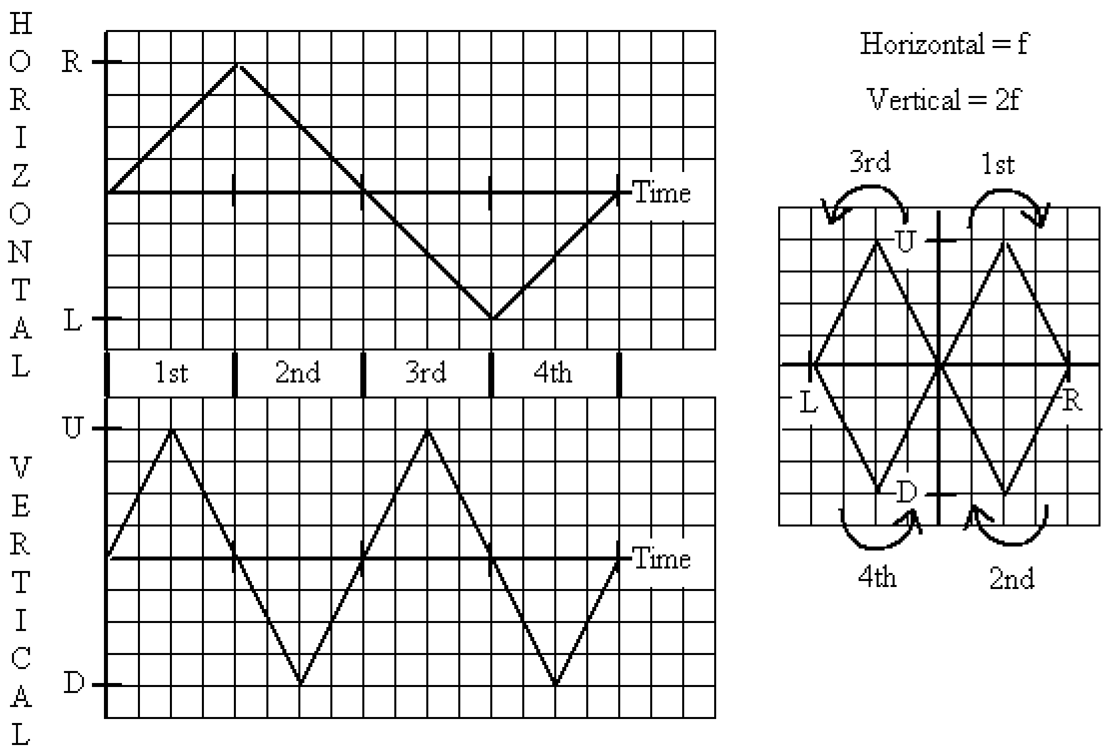
But why are we doing this? Why play the party game? Play one more game before we delve into this. See if you can understand the pattern traced out on the playing field in Fig. F-2b below. Note that the horizontal wave is the same as before. The vertical wave is a similar wave but shifted. It's not completely out of phase (180° out of phase), but halfway there. We say it's 90° out of phase. The traced pattern tells us the frequency ratio of the vertical and horizontal waves. You just count the number of points at the top and the number of points at the right. From Fig. F-2b, the answer is 1 in each case. So the frequencies are the same. In Fig. F-2a, the frequencies are also the same. There is one point at the top and at the right (same point). We plan to send two tones into the oscilloscope which is set in Lissajous mode. Then by counting the points at the top and at the right, we obtain a comparison of the two frequencies!

Fig. F-2b. Lissajous Figures: Case II.



You may be a little confused. So is everybody when they learn a new game for the first time. What should you do? Play another game. By the end of the third game, things will become clear. Refer to the third case in Fig. F-2c below. The instructions indicate that you start in the center. For the 1st quarter, you move to the right and go up and down at the same time. Then you move to the left and go down and up for the 2nd quarter. Refer to the diagram at the right to see these sections of the trip. Can you figure out the paths for the 3rd and 4th quarters?

Fig. F-2c. Lissajous Figures: Case III.



Now for the analysis. We already know the answer for the frequency comparison. The vertical wave has twice the frequency as the horizontal. The vertical wave has two complete cycles in the four quarters of our game time while the horizontal wave has one complete cycle. How can we figure this out from the game board at the right in Fig. F-2c? We compare the number of points at the top to the number of points at the right. The comparison is 2 to 1. We write this as 2:1. As you trace out the combination of vertical and horizontal oscillations, you reach the top two times for every time you reach the far right. That means your vertical frequency is twice the frequency of the horizontal.

We use Lissajous figures electronically to determine the vertical frequency (unknown) relative to our horizontal or reference frequency. However, since it's hard to lock in on the phases, the waves drift. Sometimes we get the baseball diamond for our 1:1 case (Fig. F-2b), shifting into the line of Fig. F-2a as the phases change. The shifting pattern is perceived as a rotation.

The Just Diatonic Scale

We are now ready for our big experiment. We set our oscilloscope to Lissajous mode and connect two tones to it. The reference tone *Do* is connected to the horizontal input. The tone that we will change and measure relative to the reference is connected to the vertical. It is interesting to turn the sound off and just work with the patterns. We scan the vertical until we get nice patterns. These have simple ratios like 2:1 or 3:2. For the ratio 3:2 (read as 3 to 2), we find 3 points at the top and 2 at the right side.

This indicates that the vertical travel is up and down 3 times during the time it takes to go back and forth 2 times along the horizontal. Relating to money, it's the case where your friend has 3 half-dollar coins and you have 2. The ratio is 3 to 2. You have 100 pennies worth of money and your friend has the equivalent of 150 pennies.

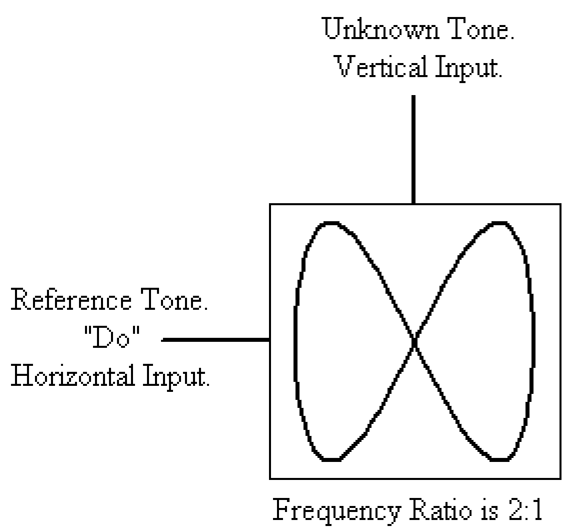
As we continue along these lines we find that simple ratios like 2:1, 3:2, 4:3, 5:4 etc. sound pleasant. We also discover that these tones are in the major scale! For example, suppose we look at the 2:1 case. We play the reference note, then the vertical input, which we know is twice the frequency as the reference. As we listen to the reference (horizontal) and then the tone on the vertical played right after the reference, we recognize the beginning of *Somewhere Over the Rainbow*. So we know we have the octave. The octave higher than *Do* is the note *Do'* and it has double the frequency of the lower note *Do*.

This is one of the discoveries of Pythagoras. However, Pythagoras experimented with strings of different lengths. We are using electronic tone generators and an oscilloscope. Pythagoras would be impressed.

Fig. F-3 illustrates our use of the oscilloscope to obtain a Lissajous figure. We send in our reference tone *Do* into the horizontal input. The unknown tone that we can vary is sent to the vertical input. We turn the knob on the oscillator of our unknown tone until we get a nice pattern.

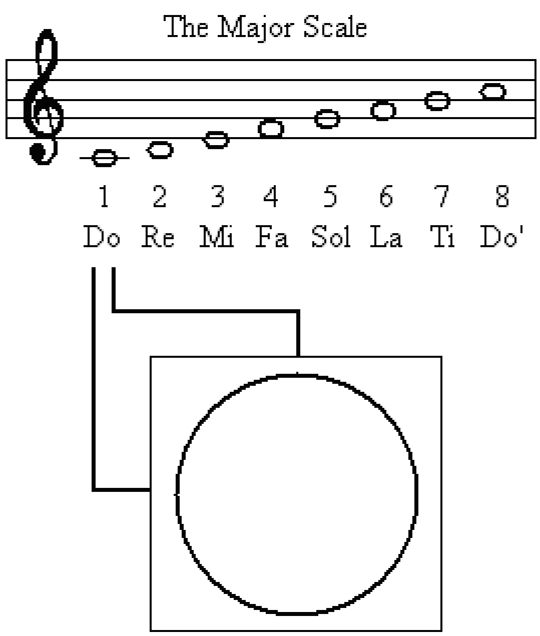
A nice pattern has been found in Fig. F-3. The pattern has rounded edges because we are using sine waves here instead of the triangle waves of Fig. F-2c. However, we can still count the number of times the wave reaches to near the "ceiling" - two times. The number of rounded extremes at the far right is just one. So we conclude that the vertical wave is twice the frequency of the horizontal wave. The ratio is 2 to 1, i.e., 2:1. We play these two tones and we hear the beginning to *Somewhere Over the Rainbow*. The interval defined by Do and the unknown tone is an octave. So the vertical tone is *Do'*.

Fig. F-3. Lissajous Figure with Oscilloscope

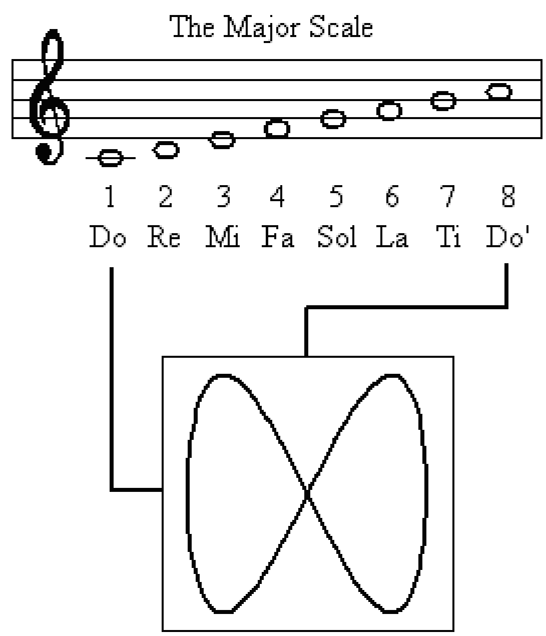


The results for several tones are given in Fig. F-4. Note that the ratio is 1:1 when both tones are the same. The unison is 1:1, the octave is 2:1, the fifth 3:2, and the fourth 4:3. These are the most pleasing tone combinations.

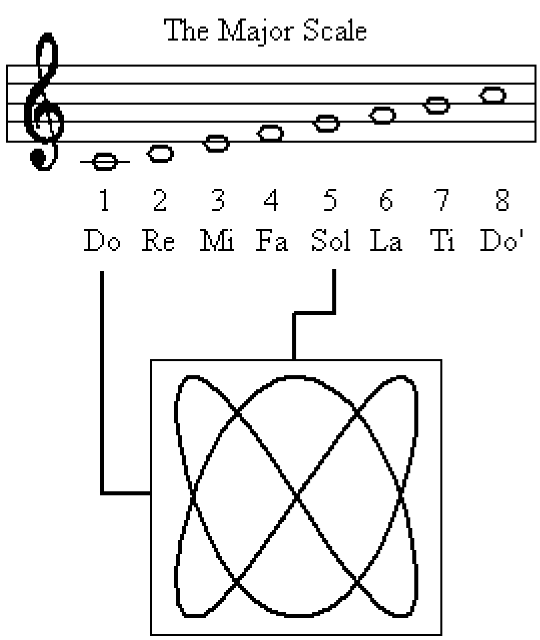
Fig. F-4. Lissajous Figures and Frequency Ratios for Some Tones of the Major Scale.



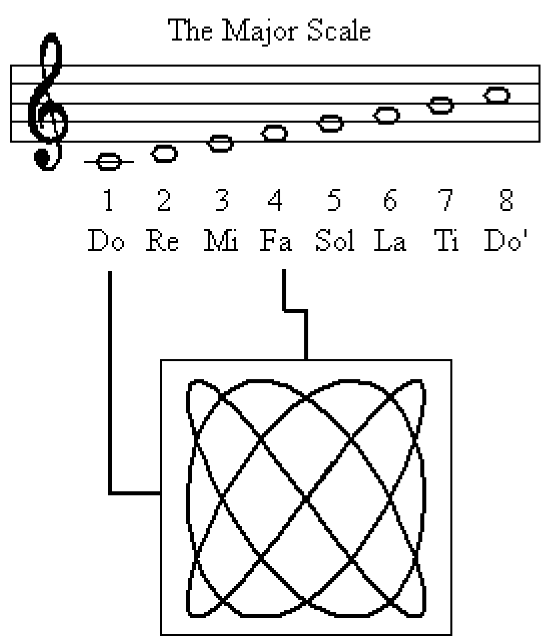
Frequency Ratio is 1:1



Frequency Ratio is 2:1



Frequency Ratio is 3:2



Frequency Ratio is 4:3

The most pleasant combinations of tones are listed in Table F-2. Tones are said be *consonant* when pleasant, and *dissonant* otherwise. However, there are degrees of consonance and dissonance. The most consonant combination is obviously when both tones are the same. This is the unison. The frequency ratio is 1:1. The next best is the octave. When the 8th tone of the scale is played with the reference tone *Do*, it sounds so pleasing that the 8th tone is also named *Do*. We use the name "Do-prime" (*Do'*) to distinguish this higher *Do* from the lower one. Many composers use octaves in writing for the piano. They are impressive when played quickly by one hand. The flashy 19th-century composer-pianist Franz Liszt often dazzled audiences with rapid octaves.

Next in line for consonance is the fifth, with a ratio of 3:2. The next best consonance is the fourth, with a ratio of 4:3. The amazing feature of Table F-2 is that the listing of consonances from best on down is grounded in mathematics. The simplest numbers are chosen to make these ratios. There is a pattern. In fact, the table suggests that the next to be investigated is the case with a ratio of 5:4. That combination is the result for the interval of a third (*Do-Mi*). You can now appreciate the wonder of Pythagoras as he discovered the mathematical foundation of musical esthetics. The pleasing or harmonious intervals are described by elegant frequency ratios.

Table. F-2. The Most Consonant Tone Combinations (Tuning to Perfect Intervals).

Tones	Interval	Ratio
Do-Do	Unison	1:1
Do-Do'	Octave	2:1
Do-Sol	Fifth	3:2
Do-Fa	Fourth	4:3

Today we do not use perfect intervals for tuning except for the octave. However, the tuning is close to the ratios given in Table F-2. The reason for this will be explained in a later chapter.

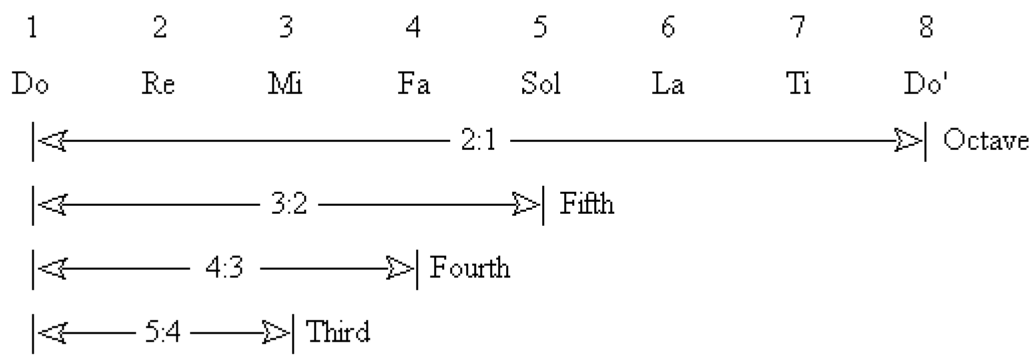
Music theory for composition and harmony incorporates the essence of Table F-2. The following analysis applies to harmony, not the melody line. The most pleasing combination consists of the same notes. However, if you just play the same note, you do not go anywhere. The next best change is to go to the octave. However, this change sounds so close to the reference (*Do*) that there still doesn't appear to be any (significant) change. The next best is movement by a fifth. This is a most pleasing change. Over the years, musicians have dressed up this change by adding related notes to support it and bring it out so to speak. Related notes when played together are called *chords*.

Chord changes by fifths serve as the basis for music theory. The musical palette of fifths is referred to as the c*ycle of fifths*. The author's jazz teacher at the University of Maryland once said (c. 1975) that 80% of popular music consists of chord (harmonic) changes that are fifths.

Frequency ratios for the most consonant intervals are illustrated in Fig. F-5. Once again, note the elegance of the mathematics. If you take the frequency for *Do* to be 100 Hz, then the octave is twice this, i.e., 200 Hz. The fifth is 150 Hz (remember our earlier discussion of the 3:2 ratio with money). The 4:3 ratio gives a frequency of 133 Hz. Can you explain why?

The 5:4 ratio gives a frequency of 125 Hz. The 5 to 4, in terms of our money example, translates to your having 4 quarters and your friend 5. You always have the dollar. You break it into 4 parts, then figure out what 5 of these parts would be. That gives 5 quarters or 125 pennies, the amount of the unknown.

Fig. F-5. Intervals and Frequency Ratios.



We would like to explain shortly, this time in some detail, how to figure out the frequencies from the ratios. We intend to do this for the entire perfect major scale.

Recall that the Greek philosopher-mathematician Pythagoras discovered the mathematics behind the consonant intervals. This was around 550 BC. Pythagoras went on to do work with musical scales, which quickly gets complicated due to playing in different keys.

The Greek astronomer Ptolemy around 150 AD developed a scale with the simplest perfect ratios. The scale presents difficulties for playing in other keys; however, it is excellent if you stick with one key.

The monk Zarlino introduced this scale for Church services in 1558. Table F-3 lists the degrees of the scale with perfect ratios for each degree. The frequencies are compared to the first degree as before. The major scale using these frequencies is called the *just diatonic scale*, *just scale*, or *just intonation*. Since the time of Bach (around 1700) we use the *equal-tempered scale*, which will be discussed in a later chapter.

Table F-3. Scale Degrees and Frequency Ratios Relative to the First Degree (*Just Scale*).

Degree in Scale	1	2	3	4	5	6	7	8
Name of Degree	Do	Re	Mi	Fa	Sol	La	Ti	Do'
Frequency Ratio	1:1	9:8	5:4	4:3	3:2	5:3	15:8	2:1

Table F-4 lists an example of specific frequencies that realizes the *just diatonic scale*. We start with 240 Hz so that the numbers come out easy. The easiest one to determine is the octave or 8th degree, which is double. But we want a systematic way to calculate these. So we will proceed in order.

The second has a frequency ratio of 9:8. The prescription to get the frequency is first to establish your reference frequency. We did that. It is 240. Now take the second number in 9:8, i.e., the 8, and divide 240 into 8 pieces. You get 30 for each of these. We want 9 of these for our tone. So 9 times 30 is 270 and we are finished. The ratio of the second degree to the first degree is 9 to 8; the second tone has 9 parts (30 per part) to the 8 parts of the reference.

For the next case, 5:4, you divide the reference 240 into 4 parts. This gives 60. Now we need 5 of these. That is 300. The third has 5 parts (60 per part) to the 4 parts of the reference. Note that the value of a part here is 60, not the same "piece size" we considered earlier.

Table F-4. Example of Specific Frequencies for the Just Diatonic Scale.

Degree in Scale	1	2	3	4	5	6	7	8
Name of Degree	Do	Re	Mi	Fa	Sol	La	Ti	Do'
Frequency Ratio	1:1	9:8	5:4	4:3	3:2	5:3	15:8	2:1
Example: f (Hz)	240	270	300	320	360	400	450	480

We now summarize these steps to determine frequencies in a compact form. For example, to find the frequency with ratio 5:4 with respect to our reference tone of 240 Hz, you simply write 5:4 as 5/4 and multiply this ratio by 240.

$$\text{Frequency} = \frac{5}{4}\ 240 = 5\ (60) = 300.$$

Use this method to verify all the frequencies in Table F-4. Do not use a calculator. You do not need a calculator to work out the examples in this text. You will understand the material better without a calculator, acquire confidence in scientific calculations, and feel better about yourself as a result.

Some Questions

These topics are discussed in class.

Describe how the 1, 5, and 4 are employed in simple songs.

Sketch the "Blues Formula."

--- End of Chapter F ---

