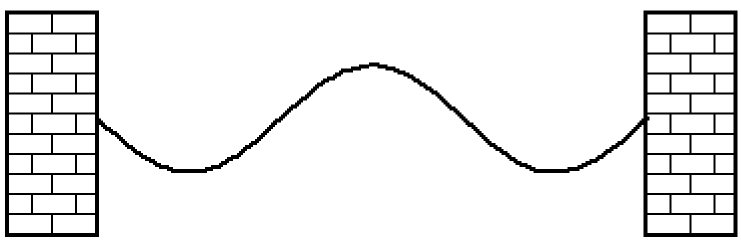
G. Strings



In the previous chapter we started with the musician's scale and found an underlying mathematical simplicity to it. Here we will start with physics and see if a natural scale arises. You might ask if we didn't do this already in the last chapter? Didn't we construct a set of tones based on simple ratios?

Almost. We found pleasant combinations in the ratios 1:1, 2:1, 3:2, 4:3, and 5:4. But what about 6:5, and 7:6? These were not in our scale. The ultimate say was the historical major scale as a given. We were delighted to see such mathematical support for the choices that went into making this scale. However, the choices were made based on historical perceptual esthetics.

Here we are going to discover a set of tones that arise naturally. We will let nature pick all the tones with no interference on our part. Nature has provided us with two very simple structures for tone production - the string and the pipe. These, along with membranes, serve as the basis for the construction of our musical instruments. We will study the strings in this chapter and the pipes in the next.

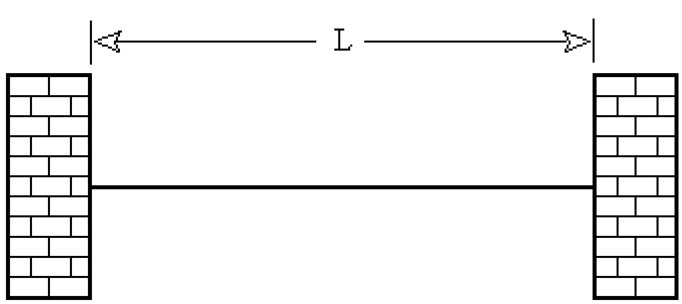
Strings are used in many instruments such as guitars, violins, the harpsichord, piano, and others. We will find that strings produce a natural set of tones. You may have heard of words like harmonics or partials. These refer to such natural tones. Some call these tones the *harmonic series*, the *harmonics*, or the *fundamental* and the *overtone series*. We might call these groups of notes the "physicist's scale" in contrast to the "musician's scale" of the previous chapter.

Just as we discovered that the musician's scale of meaningful tones has mathematical structure, we will find that the reverse is true for the mathematical "physicist's scale" or harmonic series. It has application in music, serving as a basis for musical harmonization and orchestration.

Harmonics

Consider the rope in Fig. G-1 below. Typical rope waves vibrate so slowly that we cannot hear them. However, we can see the patterns of vibration. A single rope can support a series of vibrations. We can measure the frequencies of these. The frequency ratios are most important. They will provide us with the means to establish a new set of tones, the "physicist's scale."

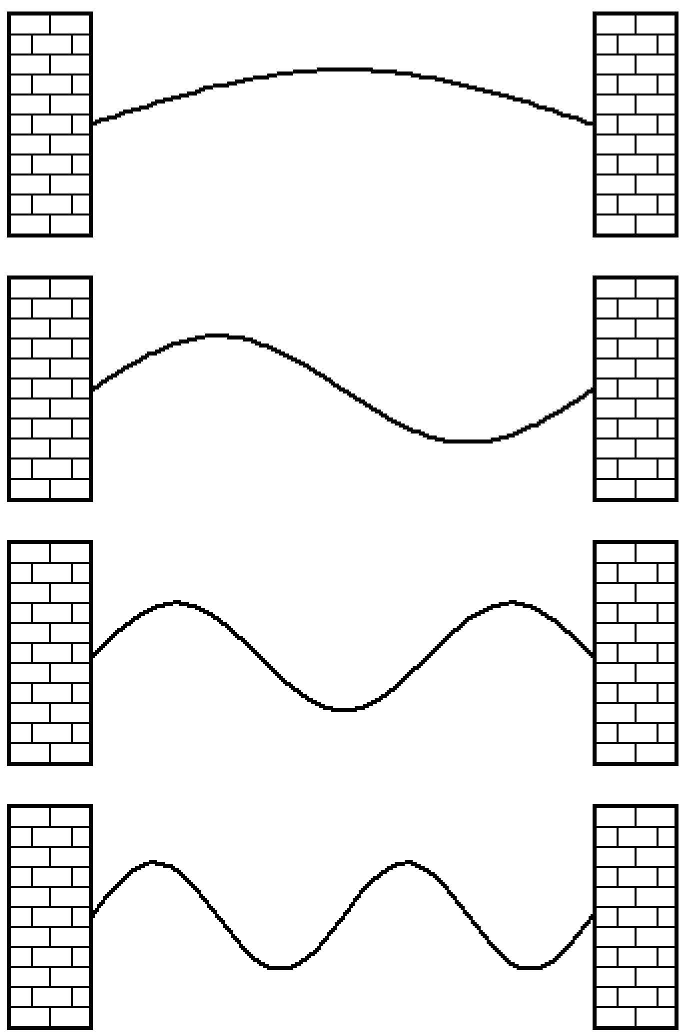
Fig. G-1. Rope or Spring of Length L in Equilibrium (No Waves).



The simplest periodic wave that a rope can support is one that we can start by pulling the middle of the rope up and letting go. The rope will vibrate up and down. We can call this the first mode of vibration for the rope. It is also the first mode of vibration for a string. See this first case in Fig. G-2. Imagine the crest swinging down into a trough and then up again. This mode is also called the *fundamental* or first harmonic.

The next mode can be obtained by pulling the first half of the rope up and the second half down, and then letting go. You will have one crest and one trough (see the second case in Fig. G-2). As the rope vibrates, the left half will go from crest to trough etc. as the right half does the opposite. Fig. G-2 illustrates the first four modes of vibration for a rope or string. These patterns can also be obtained by shaking one end of a rope or spring, while your friend holds the other end.

Fig. G-2. Standing Waves on Ropes (or Strings).



Note that the first mode consists of one half-wave, while the second consists of two half-waves, or one complete wavelength. Therefore, the wavelength of mode 2, i.e., the second harmonic, is shorter. One full wavelength fits between the walls for the second harmonic. The first harmonic has such a long wavelength that just one half the wavelength (crest or trough) fits between the walls.

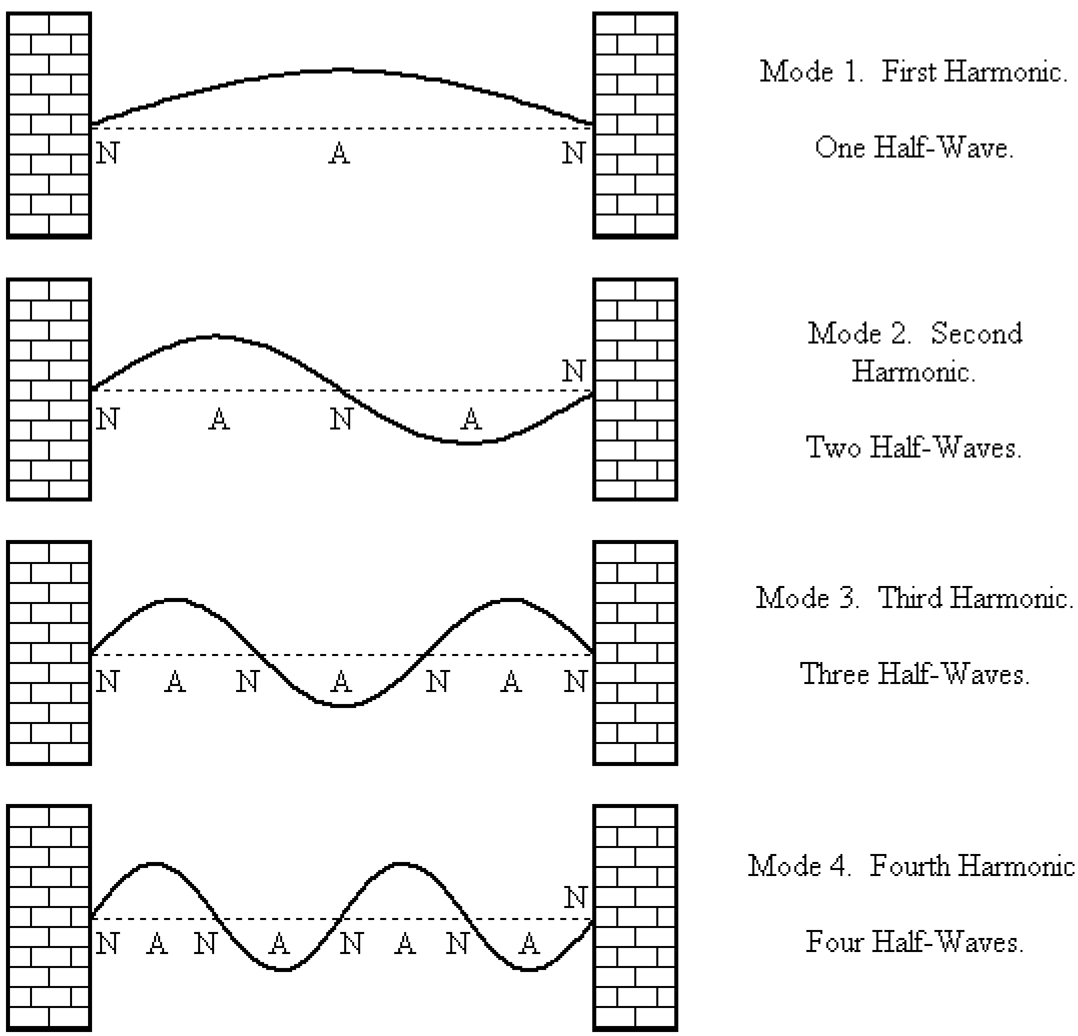
Since the speed of the waves is constant and determined by the rope or medium properties, we know that if you decrease the wavelength, the frequency increases. The second harmonic has half the wavelength of the first harmonic, therefore, the second harmonic has twice the frequency.

It is easy to measure the frequencies of rope or spring waves and directly verify the above. Can you reason in a similar fashion to determine the frequencies for the third and fourth harmonics?

Waves normally want to travel down the rope. But the rope waves hit the fixed ends tied to the brick walls. The waves reflect back and forth constantly. However, for some special frequencies the reflecting waves interfere to establish the patterns such as those in Fig. G-2 (also see Fig. G-3 below). These special waves are called *standing waves*. The patterns are essentially fixed, or "standing." There are points along the rope where the rope does not move. These points are called *nodes*. They are marked by the letter "N" in Fig. G-3. Note that the fixed ends at the walls are always nodes.

There are other points through which the rope swings to extremes in its movement. These points along the horizontal are called *antinodes* and are marked with the letter A. Sketch the fifth harmonic underneath the fourth harmonic in Fig. G-3 and indicate the nodes and antinodes.

Fig. G-3. Nodes (N) and Antinodes (A).

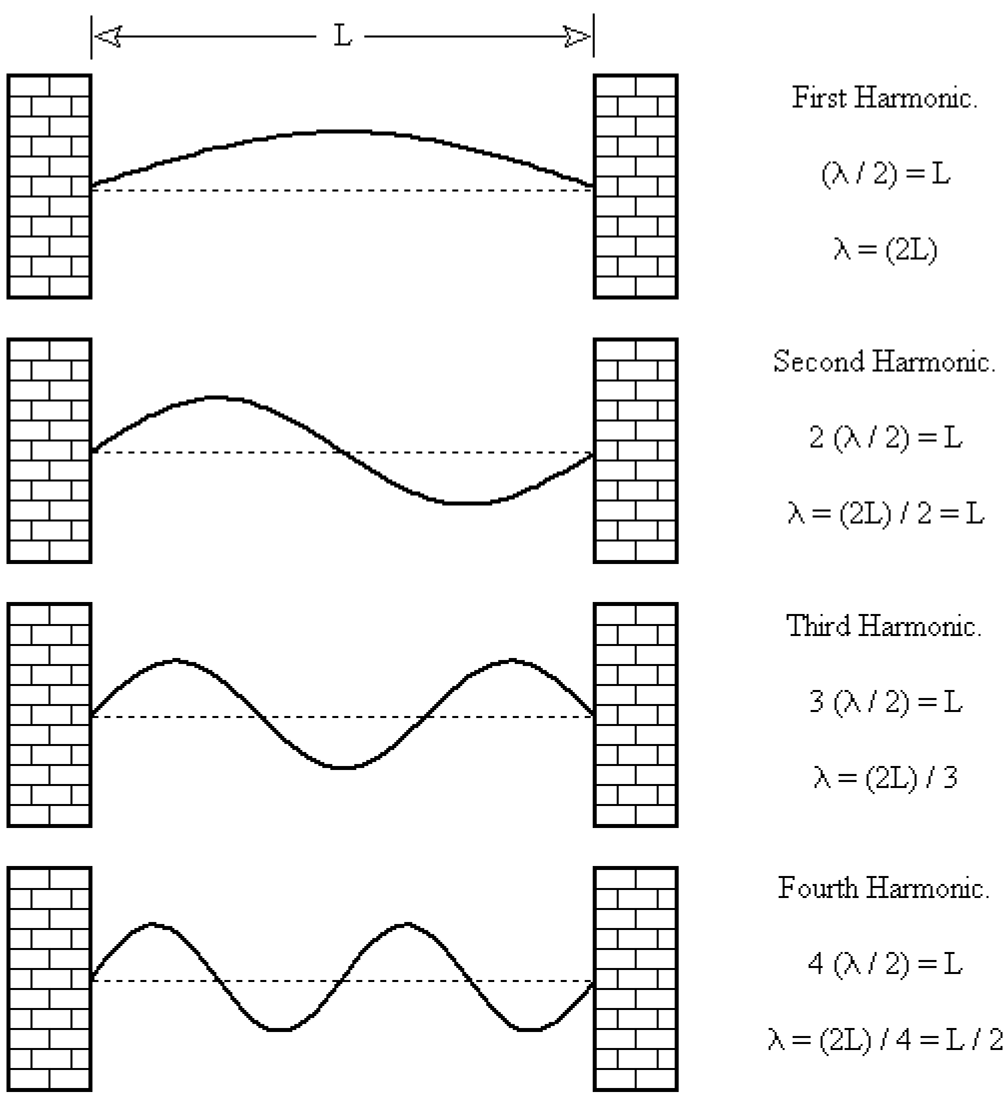


Look at the second harmonic in Fig. G-4. One complete wavelength fits nicely between the two walls. There is one crest and one trough in the snapshot of the second wave depicted in Fig. G-4. Therefore, the wavelength for the second harmonic is L, i.e., $\lambda_2 =$ L. Another easy harmonic to look at is the fourth harmonic. Here two complete wavelengths fit between the walls. The wavelength λ_4 = L/2.

A fast way to understand all of the harmonics is to note that the first harmonic has one half-wave, the second has two half-waves, the third three and so on.

The first harmonic has wavelength 2L. Don't worry that it never has a complete one between the walls. Only the second harmonic has a perfect match of one wavelength between the walls. Then as you squeeze more and more half-waves in, the wavelength must get shorter. If you squeeze in two, the wavelength shortens to 1/2 of what it was before. If you squeeze in 3 instead, the wavelength is 1/3 of the original wavelength for the first harmonic.

Fig. G-4. Wavelengths of the First Four Harmonics.



The Overtone Series

The standing waves for the string or rope are the harmonics. We list in Table G-1 the first eight harmonics. Note that the wavelengths are easily determined. As we squeeze in more and more half-waves, the wavelengths get shorter.

Compare the 5th harmonic with the 1st. The 1st harmonic has one half-wave fitting across the entire length of the string. The 5th harmonic has 5 half-waves in this same distance. Therefore, each of these must be smaller. In fact, if we can fit 5 in where before we had 1, each half-wave must be 1/5 of what we had for the 1st harmonic. Each smaller half-wave for the 5th harmonic is L/5. But since this is a half-wave, we need to multiply by 2 to get a complete wave.

The entry in Table G-1 for the wavelength of the 5th harmonic is 2L/5. To see this one more way, sketch the 5th harmonic. You should have 5 half-waves between the fixed ends. Let each half-wave be 10 cm. Then your length L is 50 cm. A complete wavelength consists of two half-waves, a crest and a trough. This complete wave is then 20 cm, or 2/5 of the length L. This is the (2L)/5 found in the table below.

Table G-1. The First Eight Harmonics

Harmonic Number	Wave-Length	Frequency	Interval from First Harmonic
1	2L	f	Unison
2	2L / 2	2f	Octave
3	2L / 3	3f	Octave + Fifth
4	2L / 4	4f	Two Octaves
5	2L / 5	5f	Two Octaves + Third
6	2L / 6	6f	Two Octaves + Fifth
7	2L / 7	7f	(Two Octaves + Minor Seventh)
8	2L / 8	8f	Three Octaves

The frequencies in Table G-1 are obtained by recalling our wave relation $\lambda f = v$. If you halve the wavelength, the frequency doubles since the product is a constant, the speed. The speed depends on the properties of the string such as its mass and tension. If your wavelength reduces to 1/3 of its original length, the frequency triples and so on. So whatever the denominator is in the wavelength column, that number multiplies our original frequency, which we take to be f.

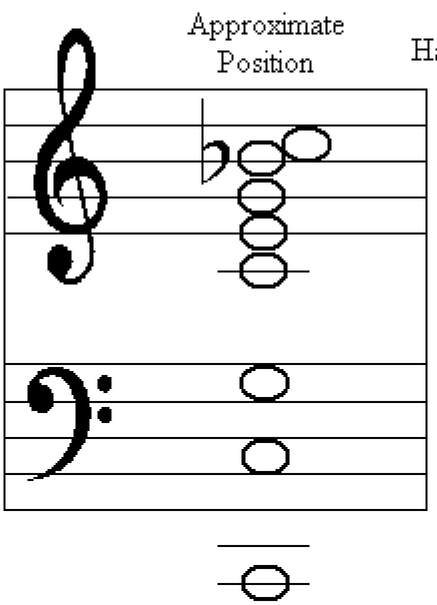
The first harmonic is also called the *fundamental*. The frequency "f" represents the frequency of the fundamental or first harmonic. The first few frequencies can also be measured directly using a spring.

The last column in Table G-1 gives the interval jump from the fundamental. Note that all the harmonics beyond the fundamental have frequencies that keep increasing. These are called *overtones* since their tones are higher or over the fundamental. The approximate positions of these tones in musical notation is given in Fig. G-5, where we have arbitrarily chosen the fundamental. The first overtone is the second harmonic H2, and it is an octave higher. Refer to the previous chapter for a review of intervals and frequency ratios. There we learned that a frequency ratio of 2:1 corresponds to an octave. The frequency ratio of H2 to H1 is 2 since H2 has frequency 2f and H1 has frequency f.

The third harmonic H3 (frequency 3f), compared to the second harmonic H2 (frequency 2f), gives us a frequency ratio of 3:2. This corresponds to a fifth. Therefore, the position of H3 relative to H1 is the interval of one octave (to get to H2) plus an additional interval of a fifth (to get from H2 to H3). To see where we are at H4, simply note that to get there from H1, we double the frequency two times. Double f and you get 2f; double again and you get 4f. We jump two octaves since every time you double the frequency, you go up an octave.

H5 relative to H4 has a frequency ratio of 5:4, which is the interval of a third. H6 compared to H4 has a ratio 6:4, which is also 3:2 by reducing. This is a fifth. So to get to H6, jump 2 octaves to get to H4, then an additional fifth to get to H6. We cannot determine H7 from information in the previous chapter. We list it in parentheses for this reason and also because this interval is very approximate anyway. Finally, to get to H8, you double the frequency of the fundamental 3 times: f to 2f, 2f to 4f, and 4f to 8f. This implies three octaves.

Fig. G-5. Fundamental (H1) and the First Seven Overtones.



Harmonic	Frequency	Interval (from H1)
H8	8f	Three Octaves
H7	7f	(Two Octaves + Minor Seventh)
H6	6f	Two Octaves + Fifth
H5	5f	Two Octaves + Third
H4	4f	Two Octaves
H3	3f	Octave + Fifth
H2	2f	Octave
H1	f	Unison

Notes are approximate since modern tuning does not employ perfect frequency ratios.

Mersenne's Laws

Mersenne listed in the early 1600s the properties of the string that determine the fundamental frequency or pitch. The first property is length. The second two properties, tension and mass, effect the speed of the waves on the string, thereby, influencing the fundamental. See Fig. G-6 below.

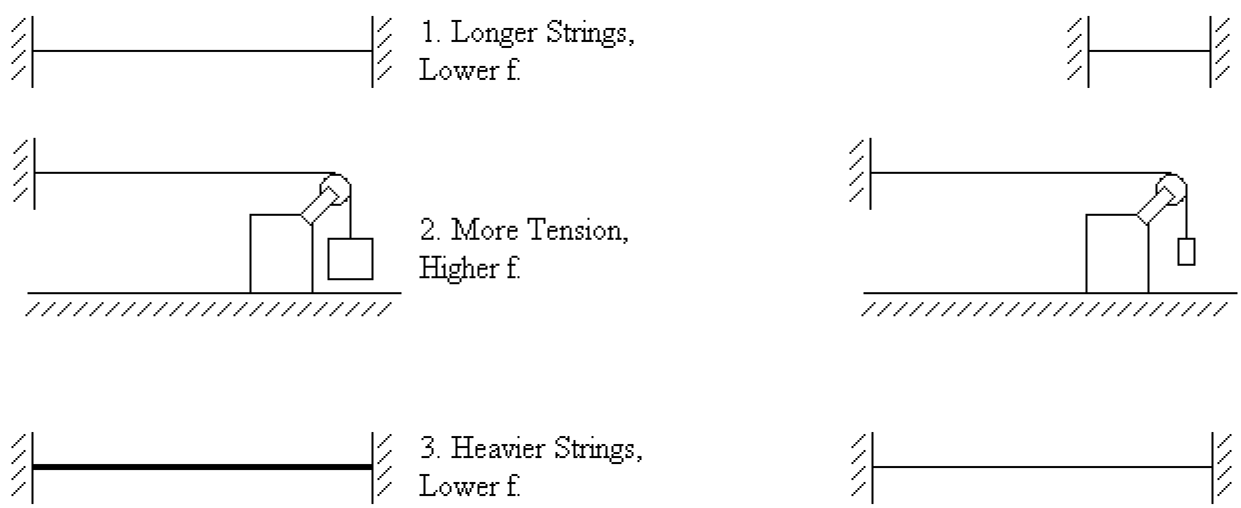
Mersenne's First Law states that the longer the string, the lower the frequency or pitch. We are accustomed to hearing the deep sounds coming from long strings on a bass.

Mersenne's Second Law states that the greater the tension in the string, the higher the frequency or pitch. When a guitar string is tightened, the pitch is raised. This is how strings are tuned. You use your hand to turn the pin that tightens strings on guitars and violins. You need to use a tuning instrument for the piano.

Mersenne's Third Law states that heavier strings result in lower frequencies or pitches. Look inside a piano. The strings down in the bass region are much thicker than the strings at the top end. You will also see that the strings vary in length. Tension is employed to hold the strings in place and give them the correct fundamental frequencies of vibration.

When strings are played, the vibration mainly consists of the fundamental. However, overtones are present in the usual complex vibrations. We will learn in a future chapter that it's the overtones that determine the timbre of a periodic wave.

Fig. G-6. Mersenne's Three Laws.



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