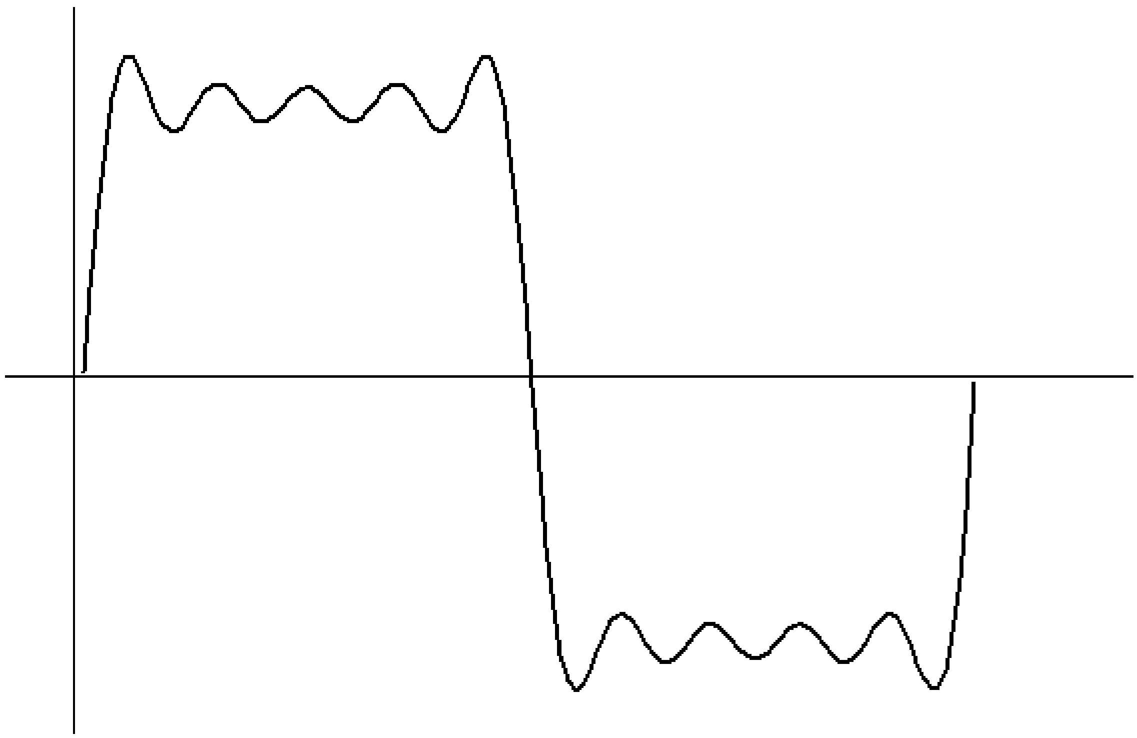
# I. Fourier Analysis



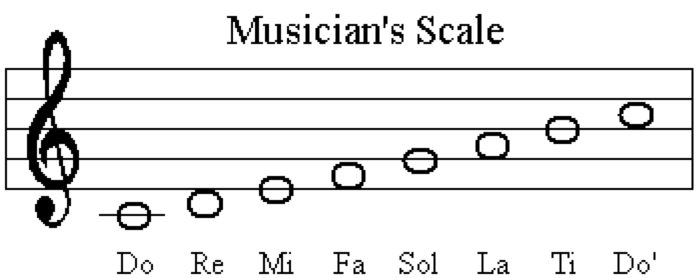
 

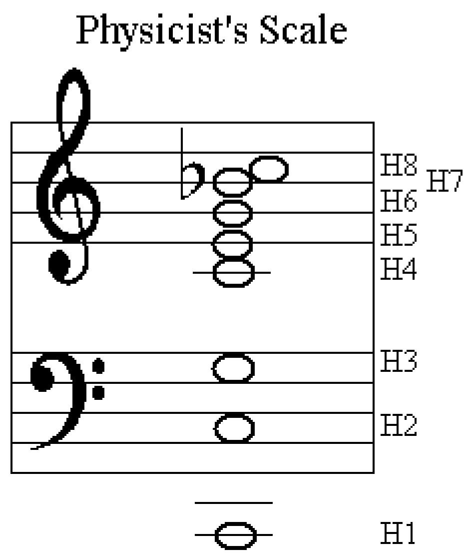
We have seen in previous chapters two sets of notes. These appear in Fig. I-1 below. The musician's scale represented is the *major scale*. What we call the "physicist's scale" comes from the modes of vibration on strings and in pipes.

First, we started with the major scale as a given. We then found that the esthetics of the scale had some underlying mathematical simplicity in terms of frequency ratios. Second, we studied the standing-patterns in strings and pipes. We discovered a very natural grouping of frequencies: f, 2f, 3f, 4f, 5f, etc., the harmonic series (the "physicist scale" in Fig. I-1). Now we will see that this scale of nature has esthetic application in music.

The harmonic series has important applications in harmony. The tones of the harmonic series blend well together. They serve as a guide when a composer wants to choose many notes to sound harmoniously when played together. Think of it this way. Use the musician's scale to pick out a melody. Then use the harmonic series to assist you in finding a group of notes (called a chord) that can be played as background to a note or subset of notes in your melody line.

Fig. I-1. The Scales of the Musician and Physicist.




A melodic line is written on music paper from left to right just like the way we read and write. The notes change in time. Refer to the musical scale at the left in Fig. I-1. The notes appear on the musical staff from left to right. On the other hand, harmonization occurs at the same point in time. Many instruments may participate in playing different notes simultaneously. These supporting notes are arranged vertically underneath the melody note since they are played together. The harmonic series in Fig. I-1 above is written in this vertical fashion. We give an example of harmonization from an orchestral excerpt from the great master Tchaikovsky in Fig. I-2. The use of the harmonic series is apparent. The excerpt is from the second movement of his *Fourth Symphony*, completed in 1878.

The excerpt in Fig. I-2 has been stripped of any musical markings for loudness. It has also been transposed slightly so that the fundamental matches that found in Fig. I-1. It is important for you to refer to Fig. I-1 and check the harmonics listed below. Please do this before continuing. The resulting sound with so many harmonics is very full and satisfying. Note the omission of the jazz-sounding seventh harmonic.

Fig I-2. Excerpt of Tchaikovsky's *Fourth Symphony* Transposed for Harmonic Analysis.

## Complex Waves

Fig. I-3 below illustrates three waves, each one getting more complex. The first, Fig. I-3a, is a sine wave. This is the simplest type of wave. It corresponds to a wave of "nature." The standing waves we find on strings and in pipe's are sine waves. The sine wave is sometimes called a pure wave. The sound of a sine wave is innocent, simple, pure in tone.

The second wave, Fig. I-3b, is a *complex periodic wave*. Any periodic waveform that is not simple (not a sine wave, i.e., non-sinusoidal) is a complex periodic wave. These waves have well-defined frequencies (pitches). Different repetitive shapes or waveforms give us the rich variety of timbres we hear.

The third wave, Fig. I-3c, is an *aperiodic complex wave*. Aperiodic complex waves do not repeat. These can be crashes, explosions, or anything else you can think of that cannot hold a steady or definite pitch.

Fig. I-3a. Sine Wave (The Simplest Periodic Wave).

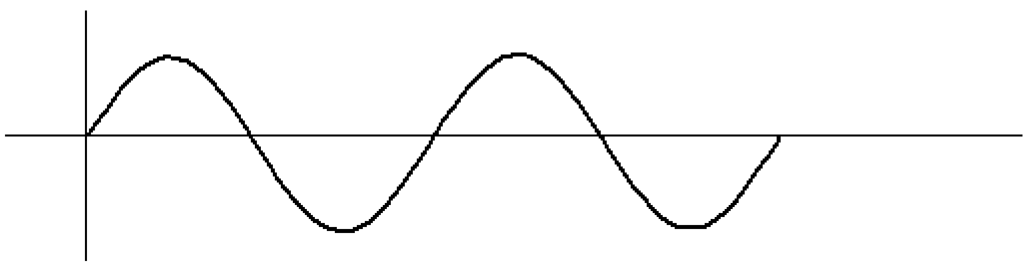


Fig. I-3b. Complex Periodic Wave (Any Periodic Wave that is Not a Sine Wave).

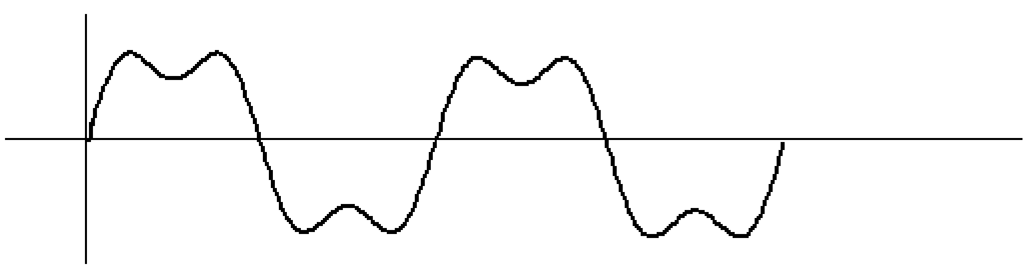
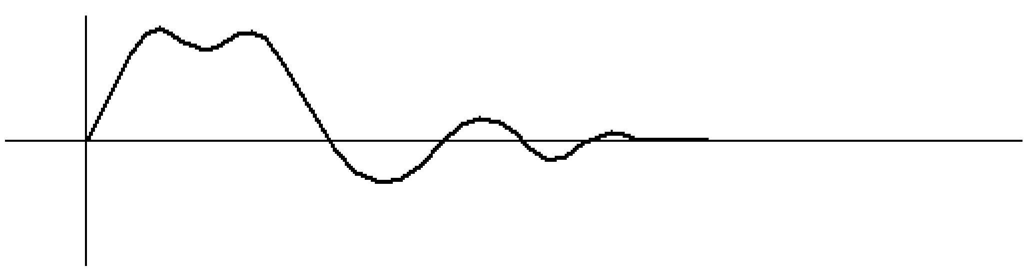


Fig. I-3c. Aperiodic Wave (Any Non-Periodic Wave).



The most interesting waves for us are the periodic waves since these have definite pitches. They include the tones of most musical instruments and the sounds of singers holding a pitch. All of the instruments we encounter in the Tchaikovsky excerpt are playing tones with well-defined frequencies or pitches. These instruments consist of three different woodwinds, the horns, and the strings. The percussive instruments in the orchestra have difficulty producing sustained tones. The piano gets help from the sustain pedal, but technically the sound is aperiodic. It dies down immediately after being produced. Timpani are drums that give a "tuned sound" for an instant. Cymbals are crashes.

There are four main sections in an orchestra: the woodwinds, brass, strings, and percussion. The first three can produce periodic tones. That covers a lot of ground. In fact, these sections of the orchestra are the ones that play most of the time. Strings come first, then woodwinds, and finally brass in order of typical use. Percussion is employed sparingly in a usual orchestral work.

So our focus on periodic waves is justified. And remember, all singers are included because they can sing a pitch and hold it. Even those of us that can't sing usually can hum a note for a few seconds. So most instruments, all singers, and the rest of us produce tones with different timbres.

Even within the same instrument category, timbre varies. The timbre of a *Stradivarius* is different from your common-variety violin. In fact, the timbre is the signature or fingerprint of the instrument. However, timbre also varies somewhat for different ranges of notes on the same instrument! In the next section you will learn how to analyze timbre.

### Fourier's Theorem

While the musician may analyze a melodic line in terms of a musical scale, the physicist analyzes timbre in terms of harmonics (which we have called the "physicist's scale"). A mathematical physicist, Baron Jean Baptiste Joseph Fourier, presented an astounding theorem in 1807, which we state below. He found that all periodic waves can be constructed from sine waves in the harmonic series!

> ***Fourier's Theorem***
>
> One can construct any periodic wave having frequency f, using sine waves with frequencies f, 2f, 3f, 4f, ... (the Harmonic Series).

The claim is that you can make any periodic wave from its harmonics. This is a very profound statement, and one which we would like to demonstrate with a specific example. We will take a square wave and challenge the theorem. How can sine waves be combined to get a square? You might be skeptical. That's good. You should be. It makes for a good scientist.

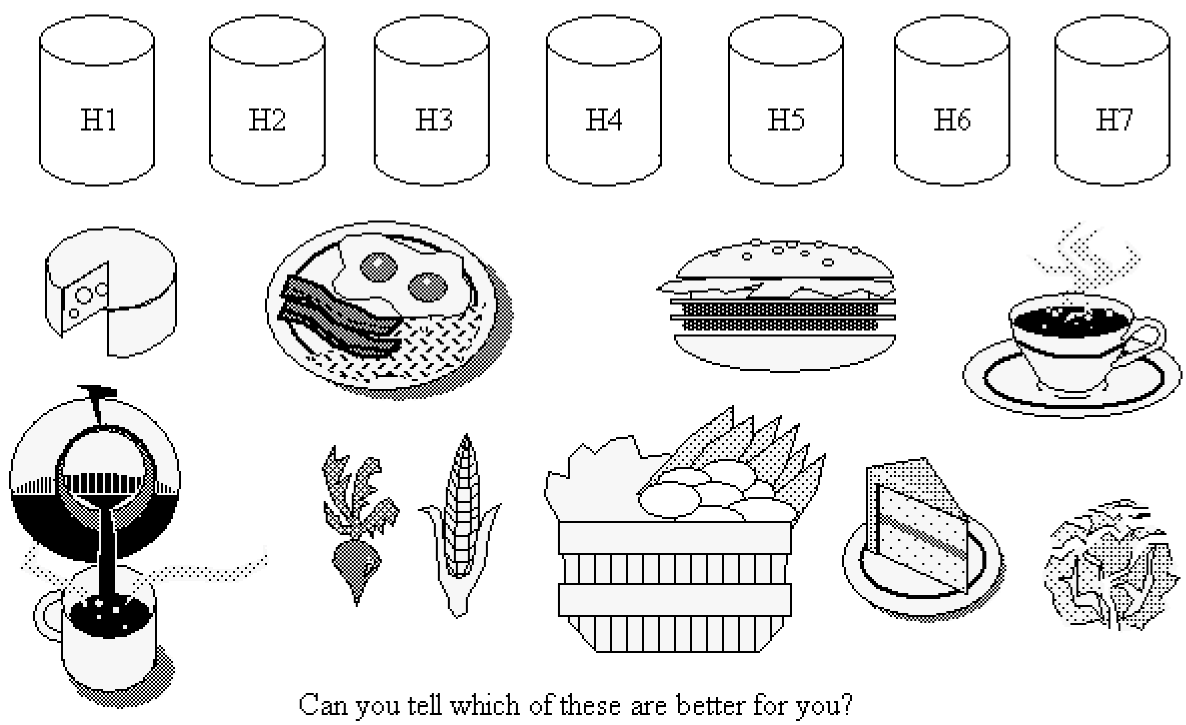
 

Suppose there was a claim that you could make any food from a set of basic ingredients. You might call the basic series of ingredients the "harmonic series of cooking." Fig. I-4 below shows such a hypothetical series of ingredients, appropriately labeled H1, H2, H3, and so on. Only seven are shown below. Imagine these in your magic cupboard. You accept any challenge of a food to cook. As you go to work, you might not have to use every ingredient in your collection. You would also use specific amounts of those ingredients needed. If you wrote out what you used and how much of each ingredient, you would call this a recipe. It would be the secret for cooking the specific meal you were asked to cook.

Similarly, when presented with a periodic wave, we can write down which ingredients, i.e., harmonics, we use and just how much of each. We mix the ingredients by simply adding harmonics. We know how to add waves; we simply add the displacements. So the recipe will tell us which harmonics to use and just what amplitude to choose for each. We will refer to an amplitude of 1 as one cup's worth. The various amounts we need of each harmonic can be called the Fourier recipe in honor of Fourier.

Actually it's called the *Fourier spectrum*. But there is one more thing a complete recipe needs to tell us. This is how to position the harmonics before adding them. How are they aligned? Are they in phase? So we need the phase relationships. But the spectrum will only supply the amplitudes since these alone essentially affect the sound of the tone. This may be starting to sound complicated, but it really isn't. The best way to convince you of this is to work out an example.

Fig. I-4. Fourier Synthesis is like Cooking!



The square wave we would like to make from sine waves is illustrated in Fig. I-5. The square wave is a periodic wave but we show one wavelength below. We purposely choose a difficult waveform, one with corners. *Fourier's Theorem* states that we can construct or synthesize this square wave from the harmonics that begin with the frequency of the periodic square wave. Let's call this frequency f. The fundamental therefore has the same frequency and thus the same wavelength of our square wave.

The method we employ in building up the square wave from the harmonics is more artistic that scientific. Physics and engineering majors learn mathematical methods to find the Fourier spectrum (recipe). Does that mean they understand it better? No. In fact, you will understand it better because you will see the square wave take shape as we go, rather than become overwhelmed by the obscurity of mathematics. The author speaks from experience. He finally understood Fourier analysis in graduate school when he did it the way you are about to see.

Fourier's Theorem directs us to the harmonics that build on the same frequency of the square wave we are trying to make. Therefore, the first harmonic is the sine wave with the same wavelength as our square wave. This first harmonic is shown in Fig. I-6. Let's use this as a reference for our amplitude and say it has amplitude 1. We go to our harmonic "cupboard" so to speak and use a full cup of our first harmonic.

It resembles our square wave a little. Doesn't it? You are probably not impressed with Fig. I-6 being a match for Fig. I-5. But we only used one harmonic (one ingredient). Things will shape up as we proceed (to cook).

Fig. I-5. Square Wave to Fourier Synthesize.

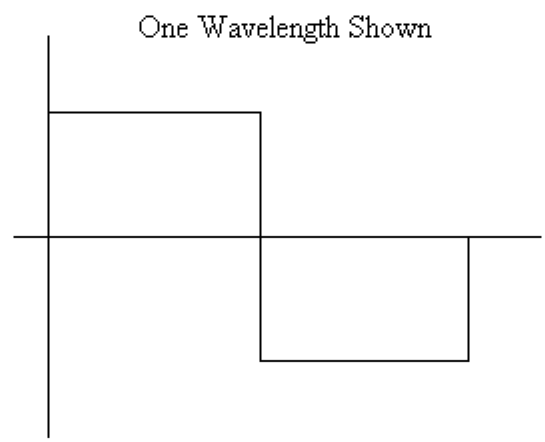



Fig. I-6. First Harmonic.

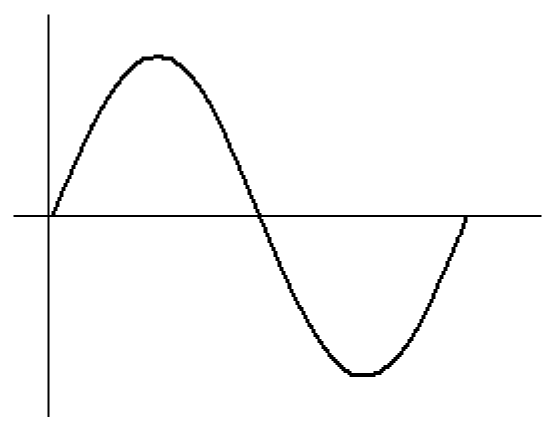


Fig. I-7. Corrections Needed for H1.

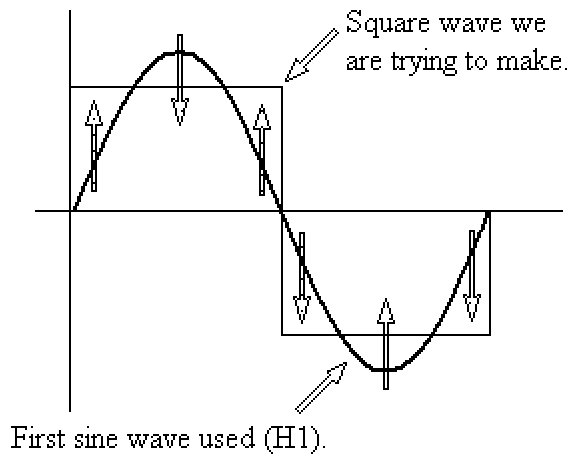



Fig. I-8. Adding H3 at 1/3 Strength to H1.

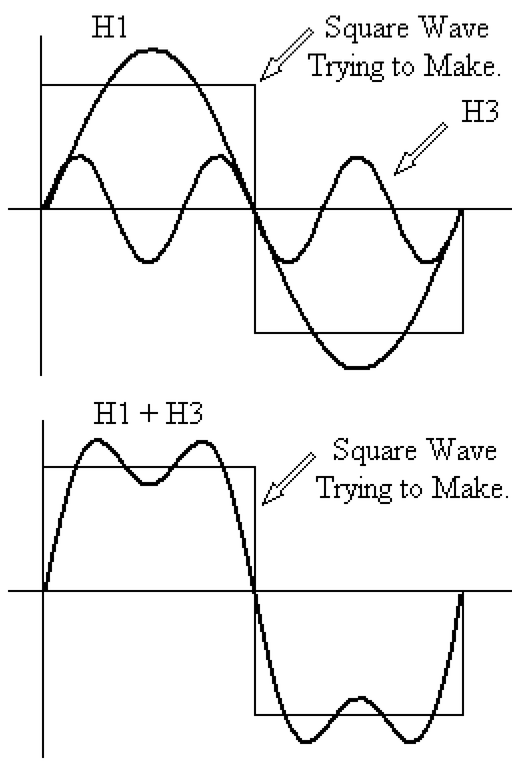



Fig. I-7 shows the shortcomings of our first harmonic H1. The square wave is included in the background so we can compare the two.

The first part of H1 (left edge) is too low. It needs to come up a bit. See the first arrow at the left in Fig. I-7. Similarly, the crest is too high. The crest needs to come down some. Then, the section just to the right of the crest needs to be taller. The trough needs similar corrections, but in reverse directions.

The vertical arrows in Fig. I-7 indicate all the major corrections necessary to improve the match with the desired square wave. The third harmonic has 3 crests and 3 troughs in just the right places to make these corrections.

See the upper diagram in Fig. I-8 for a sketch showing H1 and H3. We use H3 with an amplitude of 1/3. We might say we use the ingredient H3 with 1/3 of a cup. Note that we skip over ingredient H2. Remember, we do not have to use all the harmonics or ingredients for a specific recipe.

The result of adding H1 at full amplitude and H3 at an amplitude of 1/3 is given in the lower diagram of Fig. I-8. You might ask why an amplitude of 1/3 for H3. The recipe calls for 1/3 by trial and error. Actually, a mathematical procedure can be used to arrive at the precise value of 1/3.

Nevertheless, our sketch indicates that 1/3 is a good value for the amplitude of H3. Don't try to add the waves precisely; strive instead for a qualitative understanding of the addition. Note that H3 interferes constructively at its first crest. This shores up the left end a bit. Then note that the first trough of H3 interferes destructively with the crest of H1. It pulls the crest down. Study the effects of the addition on the other sections.

Fig. I-9. New Corrections and H5 to Assist.

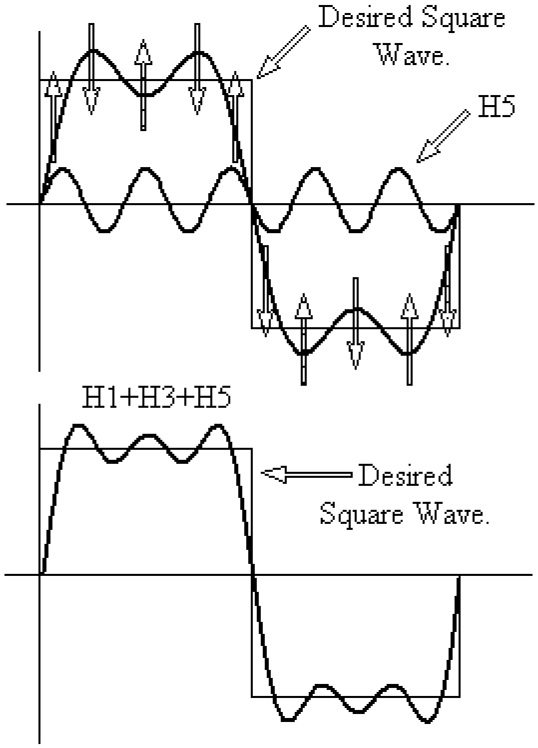
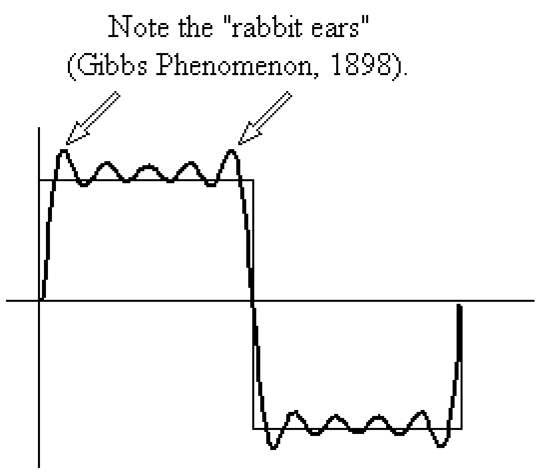



Fig. I-10. Sum of Odd Harmonics Up to H9.




We mark with vertical arrows the further corrections we need to make for our wave to look even more like a square wave. See the upper diagram in Fig. I-9. We find that we need 10 corrections, 5 up and 5 down. Note that they alternate. The fifth harmonic H5 has 5 cycles of crests and troughs. This harmonic is sketched in the upper diagram with an amplitude of 1/5. The corrections need to be gentle so we use this smaller amplitude for H5. Note that the 4th harmonic is skipped over for synthesizing a square wave.

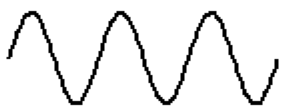
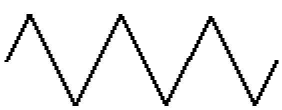
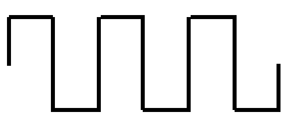
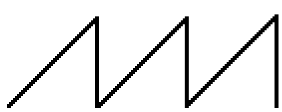
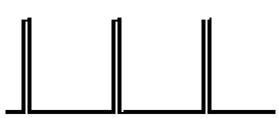
The result after adding the 5th harmonic to the sum of the fundamental and 3rd harmonic is given in the lower diagram in Fig. I-9. We are closer now in our approximation to the square wave. Observe that H1 alone has a single crest. The sum of H1 and H3 gives a "two-bump" crest and the sum of H1, H3, and H5 gives a "three-bump" crest. Notice also that the trough has these three small bumps but reversed. See if you can sketch the waveform resulting from adding in H7. This harmonic is used with an amplitude of 1/7. You should have 4 bumps in the first half of the wave. Fig. I-10 shows the result when H7 and H9 are included. The result has 5 bumps in the crest region. Note that we added 5 odd harmonics and have 5 bumps. The amplitude used for H9 is 1/9.

The sum wave begins to look more and more like a square wave. The prescription calls for using the odd harmonics, i.e., sine waves with frequencies f, 3f, 5f, 7f, 9f, and so on. The specific amounts to add (amplitudes) are 1, 1/3, 1/5, 1/7, 1/9, and so on respectively. As many more harmonics are added, the wrinkles get ironed out. The overshoot at the edges were discovered by Gibbs. They do not get ironed out; however each "rabbit ear" is squeezed shut as an infinite number of sine waves are added. The result is a perfect acoustical match.

We successfully constructed a square wave from the harmonic series. This is called *Fourier synthesis*. *Fourier analysis* is the breaking down of a periodic wave into its harmonics. Since any tone held by a singer or musical instrument is a periodic tone, it can be Fourier analyzed, usually by scientific instrumentation. The harmonics above the fundamental are referred to as overtones, as discussed earlier. Each harmonic is also called a partial (part of the complete tone).

The simple waveforms introduced earlier can be analyzed by theoretical methods of mathematical physics. The results for these simple geometrical waveforms are given in Table I-1 below (first nine harmonics). The partial-wave components (partials) are not always lined up with the fundamental for adding as in the case of the square wave. They need to be shifted left or right by various degrees, whatever the recipe calls for. The amount shifted is given by the phase. However, these are not listed since the more important ingredient in hearing is the amplitude information.

Table I-1. Five Periodic Waveforms and Their First 9 Fourier Amplitudes.

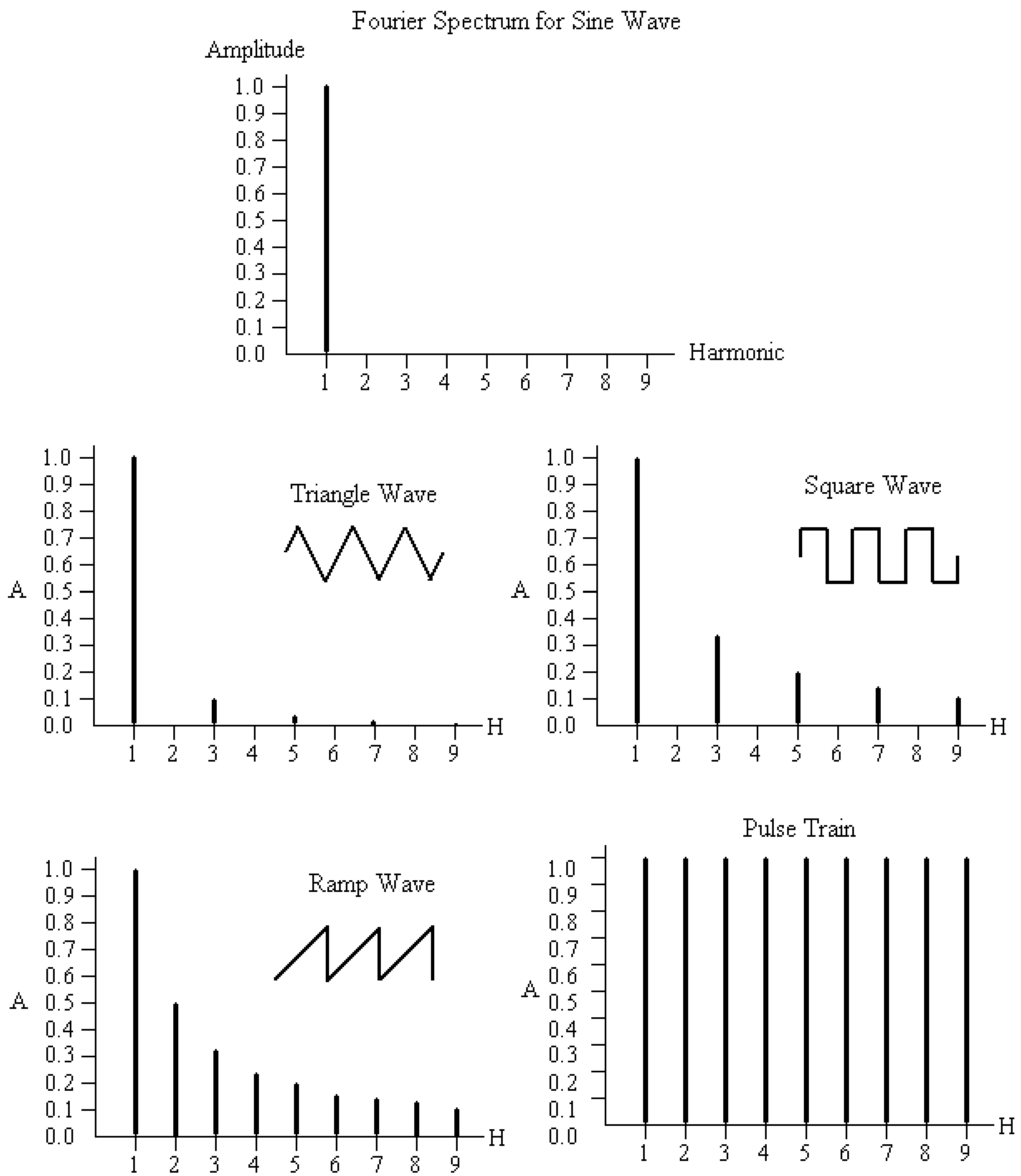
| Waveform | Shape | H1 | H2 | H3 | H4 | H5 | H6 | H7 | H8 | H9 | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Sine |  | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | Simple |
| Triangle |  | 1 | 0 | 1/9 | 0 | 1/25 | 0 | 1/49 | 0 | 1/81 | Weaker Partials |
| Square |  | 1 | 0 | 1/3 | 0 | 1/5 | 0 | 1/7 | 0 | 1/9 | |
| Ramp |  | 1 | 1/2 | 1/3 | 1/4 | 1/5 | 1/6 | 1/7 | 1/8 | 1/9 | Stronger partials |
| Pulse Train |  | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | Rich |

## Fourier Spectra

The results of Fourier analysis are most conveniently expressed in a bar graph. The plot of the relative amplitudes is often referred to as the *Fourier spectrum* for the periodic wave. The Fourier spectra for the five waveforms in Table I-1 are given in Fig. I-11. Note that the Fourier spectrum for the sine wave is just one harmonic, the fundamental or sine wave itself.

Fig. I-11. The Data in Table I-1 Expressed in Graphical Form.



# Some Questions

Fill in the circles below for the correct answers.

| Timbre | H2 | | | | H3 | | | | H4 | | | | H5 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1/4 | 1/2 | 1 | 0 | 1/9 | 1/3 | 1 | 0 | 1/16 | 1/4 | 1 | 0 | 1/25 | 1/5 | 1 |
| Sine | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ |
| Triangle | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ |
| Square | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ |
| Ramp | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ |
| Pulse Train | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ |

Check the appropriate boxes below.

| Description | Sine | Triangle | Square | Ramp | Pulse Train |
|---|---|---|---|---|---|
| No Even Harmonics | ☐ | ☐ | ☐ | ☐ | ☐ |
| Same Amount of Each Harmonic | ☐ | ☐ | ☐ | ☐ | ☐ |
| The Amplitude of H4 is 1/4 | ☐ | ☐ | ☐ | ☐ | ☐ |
| The Amplitude of H5 is 1/5 | ☐ | ☐ | ☐ | ☐ | ☐ |
| The Amplitude of H5 is 1/25 | ☐ | ☐ | ☐ | ☐ | ☐ |

--- End of Chapter I ---