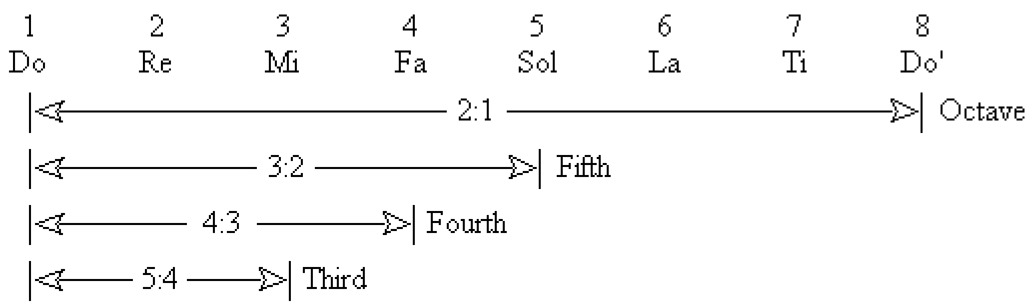
V. Musical Temperament

We saw in the last chapter how the human voice system can produce a rich variety of sounds. Earlier, we learned how engineering electronics can generate sound and modify it. We also investigated the storing of sound on media such as records, tapes, and CDs. In the final chapters we turn to the musical production of sound with traditional instruments, until the very last chapter. It is fitting that most of the final phase of this text be dedicated to that which historically has provided our current culture with the rich esthetic experience of musical art.

The Major Scale

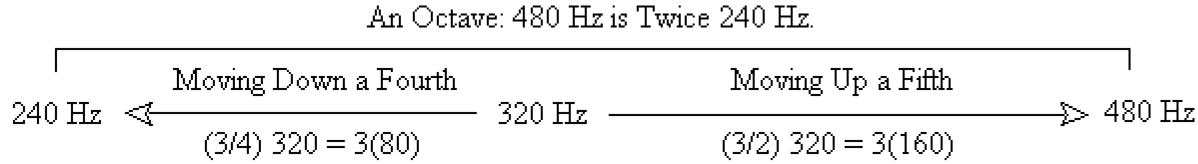
The major scale is depicted in Fig. V-1 below. The frequency ratios are indicated for a perfect octave, perfect fifth, perfect fourth, and perfect major third. These intervals provide for the most consonant combinations of tones after the unison. The octave is so close to the sound of the unison (two identical notes sounding) that we proceed to the next ratio (the 3-to-2) to serve as a foundation for a system of music theory, the *cycle of fifths*. Movement by fifths is very pleasing. The traditional way to end a piece is to move from the fifth (*dominant*) to the root (*tonic*), achieving a sense of coming home and completion. Such a harmonic change is called a *cadence*.

Fig. V-1. The *Just Major Scale* and Most Consonant Tones.



Jazz musicians employ the *cycle of fifths* often. Once the author had a heated debate with a sax player of 15 years who claimed it was the *cycle of fourths* instead. Finally the author realized that the *cycle of fourths* is in a sense the same as the *cycle of fifths*. If you move up by a fifth and call the new note *Do*, moving down by a fourth gives the *Do* that is an octave lower. Since *Do* defines the key, you have a transition to the same key in each case. Fig. V-2 illustrates this connection. To move up by a fifth (3:2), multiply by 3/2. To move down a fourth (4:3), use 3/4 instead of 4/3 (in order to get a fourth lower).

Fig. V-2. Relationship Between Moving Up a Fifth and Moving Down a Fourth.



The Twelve-Tone Scale

We are going to analyze the major scale, which is the standard eight-tone scale we become accustomed to in grade school. Our analysis will show that the "perceived jump" from note to note is not the same. Rather than rely on our ears to tell us this, we will reach this conclusion from mathematical analysis, using arithmetic. We will conclude that there is room for more notes in the major scale since some of the jumps are about twice as great as others. We can stick in an extra note here and there so that the perceptual jump from each note to the very next is the same. Recognition of this fact by hearing the notes is the experimental approach. Both methods agree. Good science requires that theory support experiment and vice versa.

Fig. V-3 takes us through the analysis step by step. First we start with the major scale. We list the perfect ratios for the 8 tones relative to the first note *Do*. This version of the major scale (the just major scale), as noted in an earlier chapter, is also called the *just diatonic scale*. We next express the frequency ratios as fractions. Note that these fractions are greater than one. A fraction is simply one number divided by another. As an example, consider the ratio 3:2. We write this ratio as 3/2, getting ready to multiply our base frequency "f." The base frequency is the frequency we choose for *Do*. Therefore, we see (3/2)f in Fig. V-3.

Next we take 240 Hz for *Do* as we did before. This makes the arithmetic easier. For the fifth, (3/2)f, the result is (3/2)240 = 3(120) = 360 Hz. These have been worked out before in the text when we first established the just diatonic scale. You might want to review that chapter at this time. The frequencies reproduced in Fig. V-3 are the same. To make our analysis even simpler, we divide each frequency by 10. Then, 240 becomes 240/10 = 24. A zero is knocked off each frequency. This particular realization of the just diatonic scale is too low to be practical, but it serves our purpose. As an exercise, start with 24 Hz and work out all the other frequencies using the appropriate ratios.

The next row compares adjacent tones. The first two frequencies (24 Hz and 27 Hz) give a ratio comparison of 27/24. The next row expresses this ratio in reduced form: 27/24 = 9/8. Note the importance of ratios. The perceived jumps in frequency are based on ratios. Remember our steps along the basilar membrane are organized by ratios (equal steps of 3.5 mm for each 2:1 frequency ratio). If our 9/8 were 8/8 instead, we would have the same note. The 9/8 has an additional 1/8 beyond unity (i.e., 1). We use this 1/8 to denote the "extra contribution," the extra part beyond 1. We then consider 1/8 and 1/9 essentially the same size. If one pie is cut into 8 pieces and a second pie cut into 9 pieces, could you tell the difference between a 1/8-size slice and a 1/9-size slice? We also replace 1/15 by 1/16 since these are even closer to being the same size.

We are now ready to draw the big conclusion. There are 5 bigger "pieces of the pie" and 2 smaller pieces. We cut the 5 big pieces in half so every new piece resulting has the same size. This introduces 5 more tones. These are the 5 black keys appearing on the keyboard for each octave. We have "derived" the black keys!

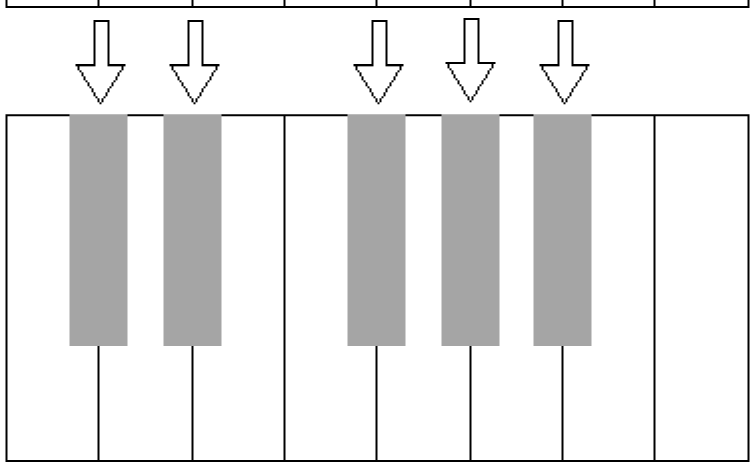
Fig. V-3. Adding the Black Keys to the Eight-Tone Scale.

The eight-tone scale.	Do	Re	Mi	Fa	Sol	La	Ti	Do'	
Just Diatonic Scale.									
Perfect ratios.	1:1	9:8	5:4	4:3	3:2	5:3	15:8	2:1	
Ratios to multiply f.	f	(9/8)f	(5/4)f	(4/3)f	(3/2)f	(5/3)f	(15/8)f	(2/1)f	
Take f = 240 Hz.	240	270	300	320	360	400	450	480	Hz
Divide by 10 for low example.	24	27	30	32	36	40	45	48	
Adjacent ratios.		27/24	30/27	32/30	36/32	40/36	45/40	48/45	
The adjacent ratios reduced.		9/8	10/9	16/15	9/8	10/9	9/8	16/15	
Extra contributions.		1/8	1/9	1/15	1/8	1/9	1/8	1/15	
Approximate extra contributions.		1/8	1/8	1/16	1/8	1/8	1/8	1/16	

There are 5 big 1/8-steps (called whole steps) and 2 small 1/16-steps, half the size (called half steps). We can split the 5 whole steps so that going from note to note is a half step in each case. This introduces 5 more notes, the "black keys."

Do	Re	Mi	Fa	Sol	La	Ti	Do'
C	D	E	F	G	A	B	C'

The last note here is considered to be the first note of the next octave. It is the starting note for the pattern to repeat an octave higher. Therefore our new scale has 12 notes.



Now we have the 7 notes from *Do* to *Ti* and 5 additional notes. This gives us a *twelve-tone scale*. We consider the note an octave higher than *Do* (i.e., *Do'*) as the beginning of the next 12 tones. For a moment, retreat to our major scale. Each step which has an extra contribution of 1/8 is called a *whole step* or *whole tone*, while the steps with the extra 1/16 contribution are called *half steps* or *semitones*. In the new twelve-tone scale, all the steps are equal. You make 12 half steps in going from *Do* to the *Do* that is an octave higher (*Do'*). To step by whole tones on the twelve-tone scale, just skip a note at each step.

Our prior restriction to the major scale limits us in playing songs since many tunes use the additional notes we have added. We can give a formula for the major scale. From your starting note you proceed to by first making a whole step. You make a whole step by skipping the very next note (whether it is a black key or white key) and land on the note after the one you skip. For a half step, you go to the very next note. The formula for the major scale is: whole-whole-half-whole-whole-whole-half (see Fig. V-4). Note that this formula consists of two whole-whole-half sections joined by a whole step or connection in the middle. The total number of steps in the scale is 7.

Centuries ago some mystics found profound meaning in the formula for the major scale. We noted that Pythagoras was a mystic and mathematician. Nearly 2000 years later, Kepler (1600) likewise felt that mystical secrets of the universe were to be found in numbers and formulas. In a sense, the professional physicist is not too far from this point of view. The secrets to understanding nature can be expressed in beautiful mathematical form. But the mystics went further. The seven steps of the scale meant much more. The number 7 was considered sacred. We see this theme often in different historical settings: the 7 days of the week, the early 7 celestial bodies of the crystalline spheres (Moon, Mercury, Venus, Sun, Mars, Jupiter, and Saturn), the 7 sacraments, the 7 colors of the spectrum, and the 7 steps of the musical scale.

Some mystics gave meaning to the two half steps in the major scale. To them, these represented a break from the usual progression of whole tones. They applied this in everyday life by saying that all efforts following from an original aim (*Do*) can get sidetracked in two key places. One is after we get started and the other is at the very end.

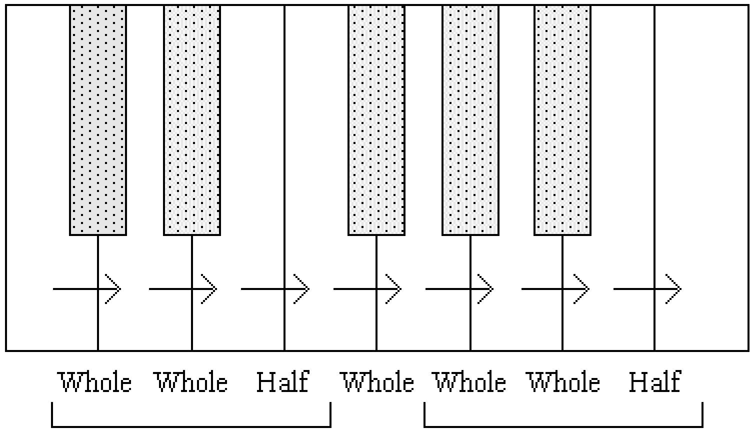
Have you every worked on a goal that proceeded smoothly for awhile (*Do*-*Re*-*Mi*) and then you reached a challenge (the half step from *Mi* to *Fa*)? Most people quit at this point. The cleaning of the room does not get completed, the term paper remains unfinished, you don't read the entire novel.

However, if you apply a conscious effort at the challenging point (*Mi*-*Fa*), you go on smoothly again for awhile (*Sol*-*La*-*Ti*) until the very end. You can still fizzle out. The modern-day version of this law is *Murphy's Law* (originating in electrical engineering): "If something can go wrong, it will." And it usually does so at the least expected places - after we just start and think things are going well, and right when we think we are about to be finished.

A scale with no half steps is the *whole-tone scale*, which Debussy liked (see Fig. V-4). Another is the *double-diminished scale*, popular in jazz improvisation. Abbreviations are used for whole (W) and half (H) steps in the example for the double-diminished scale in Fig. V-4.

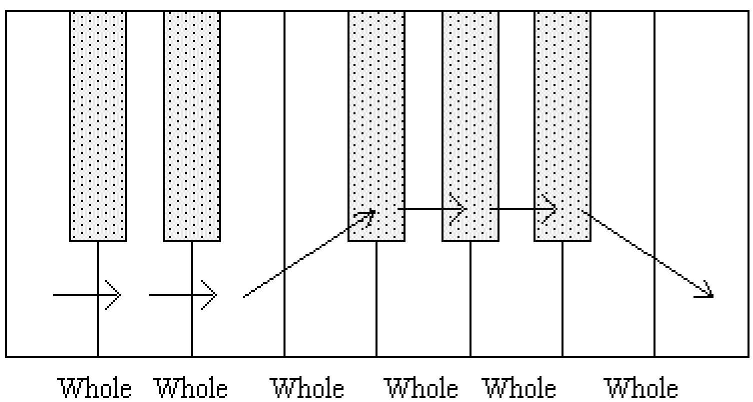
Fig. V-4. Three Formulas to Obtain Three Different Scales.

Major Scale: Do-Re-Mi-Fa-Sol-La-Ti-Do'.

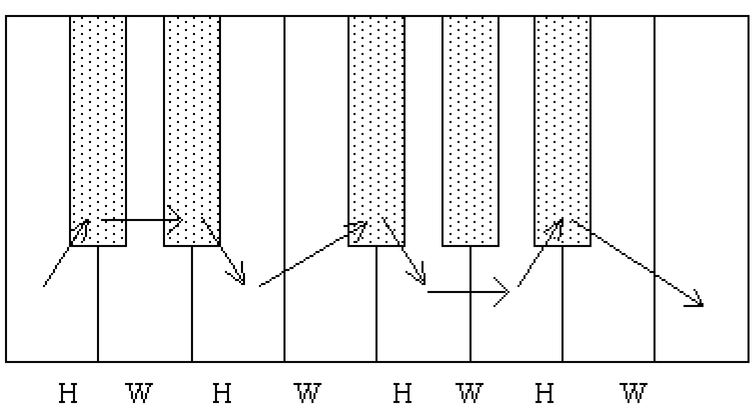


Two Similar Sections Joined by a Whole Step in the Middle.

Whole-Tone Scale, Used Often by Debussy (1862-1918).



Double-Diminished Scale, Used in Cocktail Lounges.



There are many other scales with different formulas. Go to a piano and pick out the scales we have described: the *major scale*, the *whole-tone scale*, and the *double-diminished scale*. Use Fig. V-4 for assistance. Then try the *natural minor scale* (whole-half-whole-whole-half-whole-whole). The *chromatic scale* is the scale obtained by playing all the notes (half, half, half, etc.). The modern composer Schoenberg (SHERN-berg) liked the complete *twelve-tone scale* and devised lines using the tones once and only once. These lines are called *twelve-tone rows*.

They sound strange (modern) since a note can't be used more than once in the musical line. Try writing a sentence that uses each letter of the alphabet once and only once. It's impossible. But you can write a sentence that uses all the letters of the alphabet: "The quick brown fox jumped over lazy big cats." Can you think of another that uses less than 37 letters? How about just a group of words, following the modernist Schoenberg, where the meaning can be cryptic?

Equal Temperament

Earlier in our treatment of whole steps, we considered that 1/8 and 1/9 were essentially equal. We did the same for 1/15 and 1/16 (the half steps). We would like to have more precise definitions for half steps and whole steps. Historically the difficulty with tuning to perfect ratios presented problems. If you start with a different key to play a scale, the frequency ratios are not preserved in the new key. Temperament refers to the specific choices we make for the frequencies in our scale. The uniform manner in which frequencies are chosen in *equal temperament* is described in this section.

The whole steps in our just diatonic scale have slightly different frequency ratios since really 1/8 is not exactly the same as 1/9. The equal-tempered scale solves this problem by making all half steps precisely the same in such a way that by the time you reach the octave, the frequency has doubled. The perfect-frequency ratios are given up in favor of equal-frequency ratios between adjacent tones. The only perfect interval remaining is the octave. The fifths are no longer perfectly 3:2, the fourths no longer perfectly 4:3, etc.

The remaining task is to find the magic ratio for the half step satisfying the criterion that 12 half steps give a perfect octave (2:1). This problem is identical to determining the annual interest rate (applied once yearly) needed so that your money doubles in 12 years. Each half step is analogous to each year. The growth in frequency from our example of 240 to 480 takes 12 steps. We want the growth rate to be the same from step to step. Growth in frequency is analogous to growth in money.

In banking, if you save your money, you get back the original amount plus interest. If your annual interest rate is 8% for the year, $100 earns you $8 at the end of the year. We assume interest is applied once yearly. Better deals apply interest earnings more than once a year. For example, if interest is compounded quarterly, every 3 months one applies 2% interest. This is one-fourth of the yearly rate so technically it is still 8%. However, applying the appropriate percentage more often results in a better deal for you, because your money starts to grow after 3 months.

Consider an interest rate of 10% and an initial amount of $100. After the year, the interest is $10. Our new amount is $110. We have the original $100 plus the $10 interest. We leave the $110 in for another year. The interest the following year is 10% of $110. This is $11. Note that we not only receive an interest of $10 for the $100, but an additional $1 for the $10 of interest we made the first year. We are getting interest on interest in the second year. This is good news. To get the next year amount just multiply the last amount of money by 1.1.

Table V-1 gives growth patterns for several interest rates and a starting value of $240. Values in the table are rounded off. This is unlike what banks do. They keep fractional amounts over a penny. In the table, everything is carried from year to year in the calculator, rounding off annual amounts for table entries. Which case approximates the *just-diatonic* growth pattern at the far right of the table? We can cleverly answer without looking at the table. We know from before that each half step corresponds to approximately an excessive 1/16. This fraction is about 0.06 in decimal form. That is 6% interest!

Table. V-1. Growth of $240 Until Twice the Amount is Reached.

	Interest Rates						Just-Scale Amounts (Which interest percentage comes closest?)
Time (or Step)	4%	6%	8%	10%	12%	14%	
Before 1st Year	240	240	240	240	240	240	240
After 1st Year	250	254	259	264	269	274	
After 2nd Year	260	270	280	290	301	312	270
After 3rd Year	270	286	302	319	337	356	
After 4th Year	281	303	327	351	378	405	300
After 5th Year	292	321	353	387	423	462	320
After 6th Year	304	340	381	425	474	527	
After 7th Year	316	361	411	468	531		360
After 8th Year	328	383	444	514			
After 9th Year	342	405	480				400
After 10th Year	355	430					
After 11th Year	369	456					450
After 12th Year	384	483					480

Our only demand is to match the 480 after the 12th year.

Using the 6% interest for the frequencies means that our new frequency is the old one plus 6%. To obtain the 6%-increase we multiply the old frequency by 0.06 or 1/16. The new frequency (original + 6%-increase) can be obtained by multiplying the old frequency by 1 + 0.06, i.e., 1.06. However, we see that by the time we get to twelve steps or years, the answer is a little too high. In Table V-1, the amount for 240 after 12 years is 483 instead of 480. So 1.06 is slightly too high.

We want to know that special number that we can multiply something by 12 times and arrive at twice our starting value. Let's call the special number "a." Then, multiplying 12 times, using the dot-symbol "·" for the multiplication symbol, we have

$$a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a = 2$$

Mathematicians call "a" the 12th root of 2. The answer is a little less than 1.06 as we expect. It is given below.

$$\sqrt[12]{2} = 1.059463\ldots$$

This is the number we multiply any frequency by to get the next one a half step higher. It corresponds to an interest of a little over 5.9%. It is almost 6%. How accurate in theory does this number need to be? This question is answered by perceptual psychologists. They study how close two frequencies need to be before we judge them to be the same. They research stimuli in general such as loudness, color, or taste. The difference between two stimuli where the stimuli cease to appear the same is called the *just noticeable difference* (*JND*). The just noticeable difference for frequency depends on where along the audio spectrum we are being challenged to make the match.

Notice that when high notes are played on the piano, it is more difficult to tell them apart compared to notes in the middle range. On the average, the JND for frequencies in the range of musical instruments is about 1 Hz. With a starting frequency of 1000 Hz, in order to get accuracy of 1 Hz, we can use 1.059. This gives 1059 Hz. The author once wrote a computer program to generate all the equal-tempered frequencies on the piano. One should always use the best accuracy possible for the 12th root of two, find all the frequencies, then round to the nearest hertz at the end.

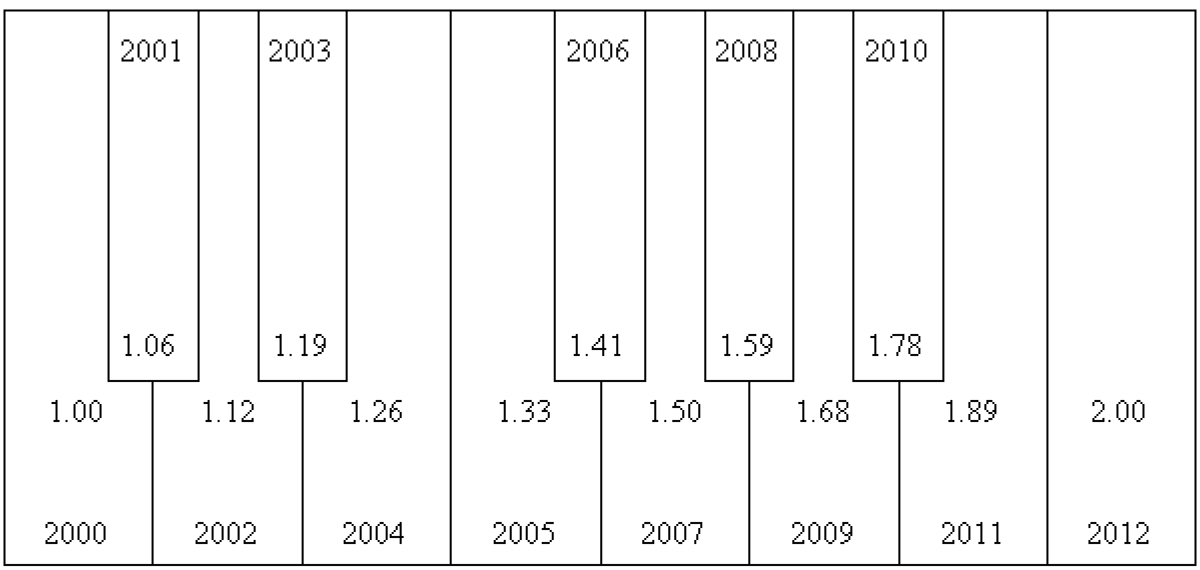
Piano tuners establish an equal temperament for the middle octave, then pretty much tune the other notes proceeding by octaves. Traditionally tuners have used their ears. Some today use electronic devices for assistance. There is debate as to which can provide the best tuning. There is no question that a well-trained tuner can do a superb job without electronic aid. Some argue that electronic devices get in the way when you strive for quality tuning.

Transposing

We would like to construct an equal-tempered set of 12 tones from which any other equal-tempered set, starting on a different first note, can be determined. So we choose 1 Hz for our first note and proceed. If you want to start with 300, then you multiply each tone of the sample set by 300. Banks do this also with loans. For example, they give values based on $1000 for home loans. If you want to borrow $105,000 and find out your interest payment, then you multiply the interest payment based on $1000 by 105.

To maintain an analogy with finance, let 1 stand for $1.00. Then, for our annual interest rate of 5.9%, we earn almost 6 cents interest after one year. The bank will give you 5 cents interest and keep the fraction of a penny. But you might say 90% of a penny is so close to a penny. Too bad! The banks round down. They make much money this way. However, when we apply the interest formula below, we round in the standard way. After one year, we have $1.06. We obtain this by multiplying our $1.00 by 1.059. The 1 in 1.059 gives us back our original dollar and the 0.059 part gives us the 5.9 cents of interest. For the 12 years or steps, we multiply by 1.059 twelve times. Actually more decimal places were used in the calculator and numbers rounded off last to get the results in Fig. V-5.

Fig. V-5. Growth of $1.00 Savings for 12 Years at 5.9463% Interest.



We can now apply Fig. V-5 to our favorite starting point, 240 Hz. Or, we might say we would like to put $240 in the bank and leave it there for 12 years. Our money will double to $480 after 12 years. We multiply each of the values in Fig. V-5 by 240 to find out how much our money is worth year by year. We "transpose" the values in Fig. V-5 to a new starting point. Musicians transpose to other keys musically by playing in different keys rather than calculating frequencies. The concept is similar.

The eight notes of the just diatonic scale are compared with equal-tempered frequencies in Table V-2. A calculator was employed that used an extremely accurate value for the 12th root of two for the equal-tempered values. Then, frequency values were rounded off to the nearest one tenth of a hertz. The starting frequency for each scale was set to 240 Hz. Since the equal-tempered scale preserves octave ratios of 2:1, the ending notes are in exact agreement. The "black keys" of the equal-tempered scale are not listed since they are not present in the just diatonic scale.

Table V-2. Just Diatonic Scale Compared to Equal-Tempered Frequencies.

Degree in Scale	1	2	3	4	5	6	7	8
Name of Degree	Do	Re	Mi	Fa	Sol	La	Ti	Do'
Just Diatonic (Hz)	240	270	300	320	360	400	450	480
Equal-Tempered (Hz)	240.0	269.4	302.4	320.4	359.6	403.6	453.1	480.0

The equal-tempered scale has a perfect ratio only at the octave. Perfect intervals in other places are lost. Remember that the JND in frequency at midrange is about 1 Hz. Therefore, the frequency difference of 0.6 Hz for the 2nd degree of the major scale is hardly noticed. However, the 2.4-Hz difference for the 3rd degree exceeds the 1-Hz tolerance. The 3rd degree on the equal-tempered scale is slightly sharp (i.e., higher in frequency) relative to a perfect major third. The 1st and 8th degrees for the just diatonic and equal-tempered scales are in perfect agreement. The 2nd, 4th, and 5th degrees are very close. The 3rd, 6th, and 7th degrees are not as close.

To compare the scales in the next octave, double every frequency in Table V-2. Consider the 6th degree. In Table V-2, the difference is 403.6 − 400 = 3.6 Hz. For the scale an octave higher, the difference between the 6th degrees is twice this: 807.2 − 800 = 7.2 Hz. Things are worse. This trend continues. However, we lose our keen frequency discrimination in the highest octave of the piano.

Musical Range

The piano is an excellent guide for studying musical range. The piano has the largest range of all instruments except for some organs. The letter names for the notes are very convenient in discussing musical range. See Fig. V-6 for the letter names of the major scale. Piano students learn them by first remembering that "C" is the key to the left of the pair of black keys. The pattern repeats on the piano. Find any place where there are two black keys grouped and there is a "C" to the immediate left. The "D" is the key that's between the two black keys, "E" is to the right. The "F" is to the left of a group of three black keys and so on. If you are not a musician, next time you are near a piano, see how fast you can hit all the "D" notes on the piano from bottom to top.

Fig. V-6. The Letter Designations for Notes.

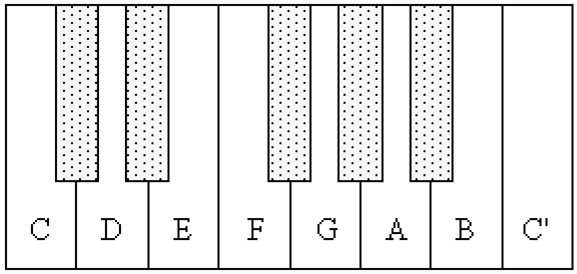


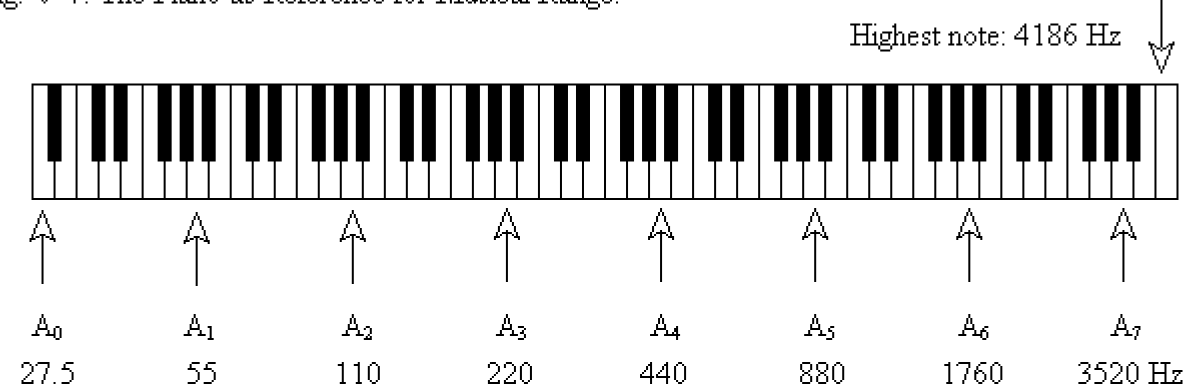
Fig. V-7 displays the entire piano keyboard. The first note (lowest) on the piano is an "A." We use the convention that calls the first "C" on the piano C_1. The note A_1 is the "A" in the major scale that starts with C_1. Therefore, we refer to the very first "A" on the piano as A_0.

Note that there are 7 complete scales starting with a "C." Since we added the black keys to the tones of the major scale, each complete scale is the chromatic scale with 12 notes. You can play a major scale starting on any of these 12 notes if you follow the formula for a major scale. There are 12 such scales, 7 for the white keys and 5 for the black keys.

Pianists spend hours learning to play these rapidly with both hands. The 7 octaves of the 12-tone chromatic scale give us 7 x 12 = 84 keys. There are three additional keys at the bottom and the sole "C" at the top. The uppermost "C" supplies the highest 12-tone scale with its resolution *Do*. Therefore, there are 84 +3 + 1 = 88 keys on the piano.

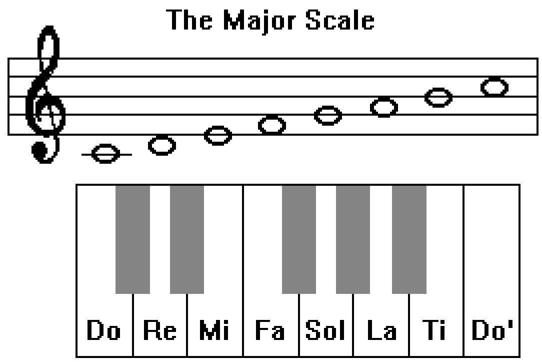
The last item remaining is to fix the frequency of one note, to get started. Equal-temperament does the rest. The standard is to set A_4 to 440 Hz. This is the note the first violinist hits on the piano when the orchestra tunes up for a piano concerto. You double 440 to get A_5 and halve 440 to get A_3 and so on in Fig. V-7. Multiplying 440 Hz by 1.059 gives the next highest key, the black key to the right of A_4. We refer to this key as $A^{\#}_4$. The symbol "#" is called a sharp symbol. We say "A-sharp-4." Dividing 440 by 1.059 gives us the black key to the left of A_4, called $A^b{}_4$ or "A-flat-4." Note that $A^b{}_4$ is also the same as $G^{\#}_4$. *Sharp* means move one half step to the right; *flat* means move one half step to the left. The piano frequencies range from 27.5 Hz (A_0) to 4186 Hz (C_8).

Fig. V-7. The Piano as Reference for Musical Range.



An Exercise

What are the frequencies for the "just diatonic" major scale if the first one is 120 Hz? Be sure to be able to work these out using the ratio for each of the intervals.



Part	Degree of Scale	Name of Degree	Frequency (Hz)
a	1	*Do*	120
b	2	*Re*	
c	3	*Mi*	
d	4	*Fa*	
e	5	*Sol*	
f	6	*La*	
g	7	*Ti*	
h	8	*Do'*	

--- End of Chapter V ---

