Theoretical Physics
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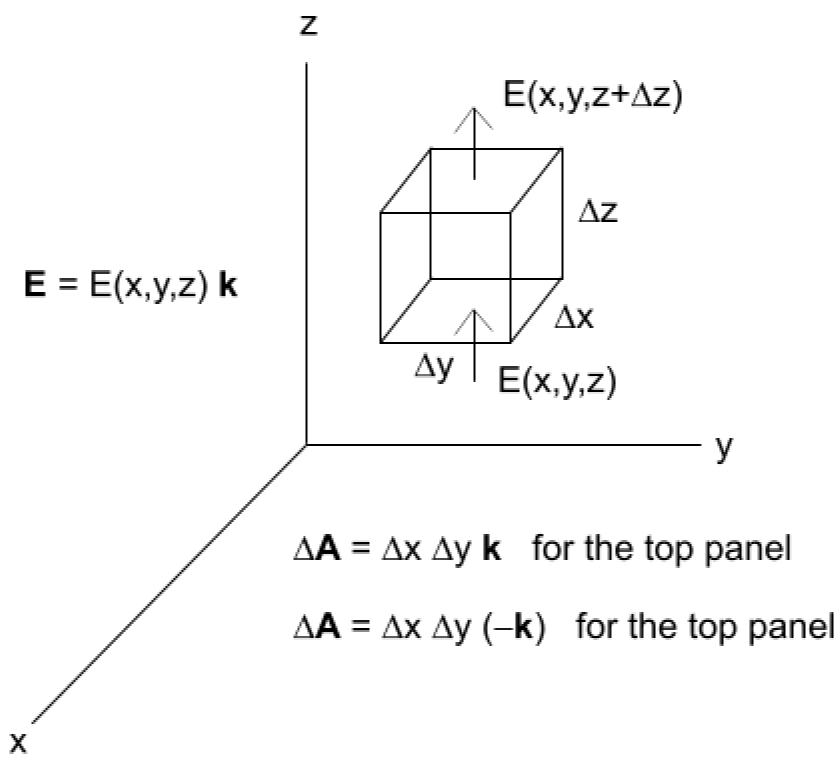
Chapter E Notes. Differential Form for the Maxwell Equations

E1. The Divergence Theorem

We are going to derive two important theorems in vector calculus in this chapter. The first one is the Divergence Theorem. We consider a vector field **E** and proceed to do a closed surface integral of this field.

$$\oiint \vec{E} \cdot \overrightarrow{dA}$$

You recognize this as the left side of our first Maxwell equation. The vector field can be any vector field. To simplify, we will pick the field to be in the z direction.



This way, it is easier to understand the basic idea. We can easily generalize to the case where the vector field has all 3 components.

We will do the surface integral over this small finite cube. Then we will take limits to shrink the cube to an infinitesimal cube.

The result will be the divergence theorem. To remind ourselves that **E** is up, we use the z subscript for E:

$$E = E_z(x, y, z) \; .$$

$$\oiint \vec{E} \cdot \overrightarrow{dA} \Rightarrow E_z(x, y, z + \Delta z)\Delta x \Delta y - E_z(x, y, z)\Delta x \Delta y$$

Note the minus sign at the bottom surface because **E** points up and the Δ**A** points down there. Refer to the figure. On the four vertical side panels the **E** field skims the surfaces so that the dot product with each of those surface elements gives zero.
So we have

$$\oiint \vec{E}\cdot\overrightarrow{dA} \Rightarrow \left[E_z(x,y,z+\Delta z)-E_z(x,y,z)\right]\Delta x\Delta y .$$

At this point, the right side is a surface integral. Now comes the trick. The partial derivative $\frac{\partial E_z}{\partial z}$ is almost staring us in the face. So set up the partial derivative $\frac{\partial E_z}{\partial z}$ and prepare to integrate it with respect to dz, which does not change anything.

$$\oiint \vec{E}\cdot\overrightarrow{dA} \Rightarrow \frac{\left[E_z(x,y,z+\Delta z)-E_z(x,y,z)\right]\Delta x\Delta y\Delta z}{\Delta z}$$

This promotes a surface integration to a volume integration when we take the limits to get differentials.

$$\oiint \vec{E}\cdot\overrightarrow{dA} = \iiint_V \frac{\partial E_z}{\partial z}\,dxdydz .$$

For the general case with

$$\vec{E} = E_x(x,y,z)\,\hat{i} + E_y(x,y,z)\,\hat{j} + E_z(x,y,z)\,\hat{k} ,$$

we have

$$\oiint \vec{E}\cdot\overrightarrow{dA} = \iiint_V \left[\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}\right]dxdydz .$$

Now it is convenient to define the operator which we call the del operator:

$$\nabla \equiv \frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k} \text{ so that } \nabla\cdot\vec{E} = \frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z} .$$

Then we have the nice notation $\oiint \vec{E}\cdot\overrightarrow{dA} = \iiint_V \nabla\cdot\vec{E}dxdydz$ and finally

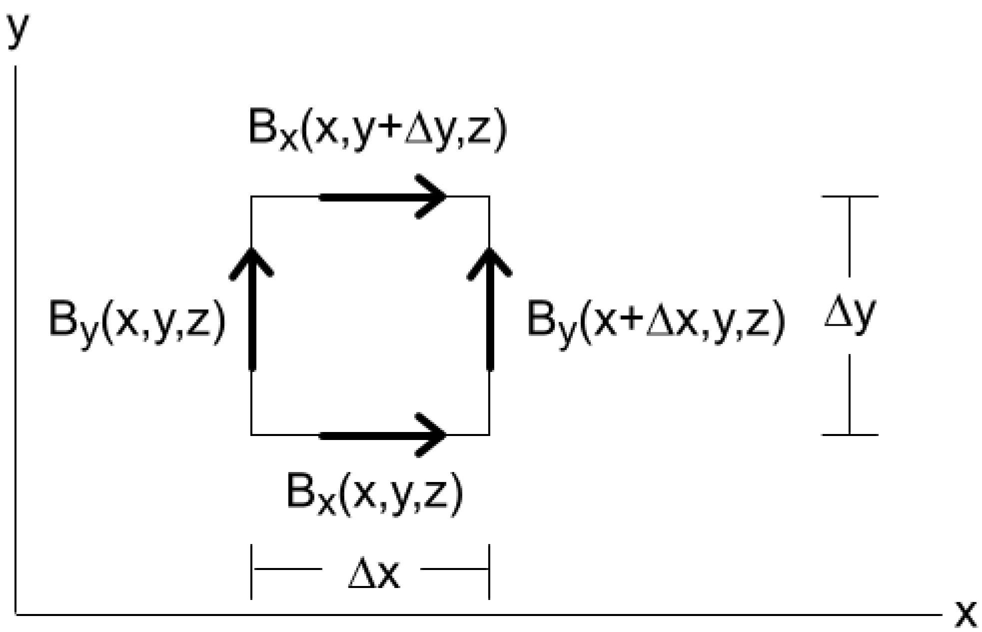
$$\oiint \vec{E}\cdot\overrightarrow{dA} = \iiint_V \nabla\cdot\vec{E}dV , \text{ where } dV = dxdydz .$$

E2. Stoke's Theorem

We consider a vector field **B** and proceed to do a closed line integral of this field.

$$\oint \vec{B} \cdot \vec{dl}$$

You recognize this as the left side of one of our Maxwell equations. The vector field can be any vector field. To simplify, we will pick the field to be in the x-y plane.



$$\oint \vec{B} \cdot \vec{dl} =>$$

$$B_x(x,y,z)\Delta x + B_y(x+\Delta x,y,z)\Delta y - B_x(x,y+\Delta y,z)\Delta x - B_y(x,y,z)\Delta y$$

$$= \left[B_y(x+\Delta x,y,z) - B_y(x,y,z) \right]\Delta y - \left[B_x(x,y+\Delta y,z) - B_x(x,y,z) \right]\Delta x$$

$$= \frac{\left[B_y(x+\Delta x,y,z) - B_y(x,y,z) \right]}{\Delta x}\Delta x\Delta y - \frac{\left[B_x(x,y+\Delta y,z) - B_x(x,y,z) \right]}{\Delta y}\Delta x\Delta y$$

This trick lets us promote the line integral to a surface integral. The derivative and integral for the extra variable does not change things.

This leads from $\oint \vec{B} \cdot \vec{dl} =>$

$$= \frac{\left[B_y(x + \Delta x, y, z) - B_y(x, y, z) \right]}{\Delta x} \Delta x \Delta y - \frac{\left[B_x(x, y + \Delta y, z) - B_x(x, y, z) \right]}{\Delta y} \Delta x \Delta y$$

to

$$\oint \vec{B} \cdot \vec{dl} = \iint_A \left[\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right] dxdy$$

Do you recognize the cross product arrangement? Consider

$$\vec{A} \times \vec{B} = \hat{i}(A_y B_z - A_z B_y) + \hat{j}(A_z B_x - A_x B_z) + \hat{k}(A_x B_y - A_y B_x).$$

Now take the A vector as the del vector operator:

$$\nabla \times \vec{B} = \hat{i}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \hat{j}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \hat{k}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right).$$

We have the z component:

$$\oint \vec{B} \cdot \vec{dl} = \iint_A \left[\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right] dxdy = \iint_A (\nabla \times \vec{B})_z \, dxdy$$

$$\oint \vec{B} \cdot \vec{dl} = \iint_A (\nabla \times \vec{B})_z \, \hat{k} \cdot \hat{k} \, dA$$

$$\oint \vec{B} \cdot \vec{dl} = \iint_A (\nabla \times \vec{B}) \cdot d\vec{A}$$

This is Stoke's Theorem.

E3. The Maxwell Equations in Differential Form

We will now transform the integral forms of the Maxwell equations into differential forms.

$$\oint\!\!\!\oint \vec{E} \cdot \overrightarrow{dA} = \frac{Q}{\varepsilon_0}$$

$$\oint\!\!\!\oint \vec{B} \cdot \overrightarrow{dA} = 0$$

$$\oint \vec{B} \cdot \overrightarrow{dl} = \mu_0 i + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

$$\oint \vec{E} \cdot \overrightarrow{dl} = -\frac{d\Phi_B}{dt}$$

1. The First Maxwell Equation

$$\oint\!\!\!\oint \vec{E} \cdot \overrightarrow{dA} = \frac{Q}{\varepsilon_0}$$

Express the left side using the Divergence Theorem:

$$\oint\!\!\!\oint \vec{E} \cdot \overrightarrow{dA} = \iiint_V \nabla \cdot \vec{E}\, dV$$

Express the right side with the volume charge density

$$\frac{Q}{\varepsilon_0} = \iiint_V \frac{\rho}{\varepsilon_0} dV$$

The more rigorous analysis leads us to write

$$\iiint_V (\nabla \cdot \vec{E} - \frac{\rho}{\varepsilon_0})\, dV = 0 .$$

Then we state that since the volume integration is arbitrary, i.e., we can take different volumes, the integrand must vanish to make the equation true in general.

Arbitrary volumes mean that the following

$$\iiint_V (\nabla \cdot \vec{E} - \frac{\rho}{\varepsilon_0})\, dV = 0$$

implies

$$\nabla \cdot \vec{E} - \frac{\rho}{\varepsilon_0} = 0 ,$$

which leads to

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} .$$

This the differential form for Gauss's Law, which in turn is equivalent to Coulomb's Law.

2. The Second Maxwell Equation

This one is easy after doing the first. Since

$$\oint\!\!\!\oint \vec{E} \cdot \overrightarrow{dA} = \frac{Q}{\varepsilon_0} \text{ becomes } \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} ,$$

$$\oint\!\!\!\oint \vec{B} \cdot \overrightarrow{dA} = 0 \text{ becomes } \nabla \cdot \vec{B} = 0 .$$

No magnetic field lines can originate at a point such that a next flux pierces out of the enclosed surface. This is a most elegant statement that there are no magnetic monopoles. The magnetic field tends to loop and the presence of a north and south pole for a magnet means we have a cancellation effect. There is no such thing as magnetic charge, at least so far as we know.

3. The Third Maxwell Equation

What about this one?

$$\oint \vec{B} \cdot \overrightarrow{dl} = \mu_0 i + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

We use Stoke's theorem for the left side.

$$\oint \vec{B} \cdot \overrightarrow{dl} = \iint_A (\nabla \times \vec{B}) \cdot \overrightarrow{dA}$$

Then we need to express the right side $\mu_0 i + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$ as an area integral. We use the definition of the current density. If you forgot about this from your intro physics course, we are led to it here. The mathematics guides us and suggests the following definition.

$$i = JA \quad \text{and in general} \quad i = \iint_A \vec{J} \cdot \overrightarrow{dA}$$

The flux Φ_E is no problem because an area is involved in its definition:

$$\Phi_E = EA \quad \text{and in general} \quad \Phi_E = \iint_A \vec{E} \cdot \overrightarrow{dA} .$$

Putting this all together:

$$\iint_A (\nabla \times \vec{B}) \cdot \overrightarrow{dA} = \mu_0 \iint_A \vec{J} \cdot \overrightarrow{dA} + \mu_0 \varepsilon_0 \frac{d}{dt} \iint_A \vec{E} \cdot \overrightarrow{dA} .$$

We move the derivative inside the integral since the integration is over area and has nothing to do with time. We write as a partial derivative as E depends on x, y, z, and t.

$$\iint_A (\nabla \times \vec{B}) \cdot \overrightarrow{dA} = \mu_0 \iint_A \vec{J} \cdot \overrightarrow{dA} + \mu_0 \varepsilon_0 \iint_A \frac{\partial \vec{E}}{\partial t} \cdot \overrightarrow{dA}$$

We rewrite

$$\iint_A (\nabla \times \vec{B}) \cdot \overrightarrow{dA} = \mu_0 \iint_A \vec{J} \cdot \overrightarrow{dA} + \mu_0 \varepsilon_0 \iint_A \frac{\partial \vec{E}}{\partial t} \cdot \overrightarrow{dA}$$

as

$$\iint_A \left[(\nabla \times \vec{B}) - \mu_0 \vec{J} - \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right] \cdot \overrightarrow{dA} = 0$$

Since the surface area chosen is arbitrary, the integrand must vanish to make this true in general. This gives us the third Maxwell equation.

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

4. The Fourth Maxwell Equation

The last Maxwell Equation is easy since it is similar and simpler than the third. Since

$$\oint \vec{B} \cdot \overrightarrow{dl} = \mu_0 i + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} \quad \text{becomes} \quad \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t},$$

$$\oint \vec{E} \cdot \overrightarrow{dl} = -\frac{d\Phi_B}{dt} \quad \text{becomes} \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

The Maxwell Equations in Integral Form (left) and Differential Form (right)

$$\oiint \vec{E} \cdot \vec{dA} = \frac{Q}{\varepsilon_0}$$

$$\oiint \vec{B} \cdot \vec{dA} = 0$$

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 i + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

$$\oint \vec{E} \cdot \vec{dl} = -\frac{d\Phi_B}{dt}$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

E4. Uses of the del Operator

$$\nabla \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

1. The Gradient

When the del operator acts on a scalar function $\phi(x, y, z)$, we get a vector function called the gradient.

$$\nabla \phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

PE1 (Practice Problem). Calculate the gradient for each of the following.

$$f(x, y, z) = x^2 + y^2 + z^2$$

$$g(x, y, z) = 2xy + yz^2$$

$$h(x, y, z) = 2x + 3y^2 + \sin z$$

2. The Divergence

When the del operators acts on a vector field $\vec{A}$ as a dot product, you have the divergence.

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

PE2 (Practice Problem). Calculate the divergence for each of the following.

$$\vec{A} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

$$\vec{B} = x^2\,\hat{i} + y^2\,\hat{j} + z^2\,\hat{k}$$

$$\vec{C} = \cos x\,\hat{i} + \sin y\,\hat{j}$$

3. The Curl

When the del operators acts on a vector field $\vec{A}$ as a cross product, you have the curl.

$$\nabla \times \vec{A} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

PE3 (Practice Problem). Show that

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

PE4 (Practice Problem). Calculate the curl for each of the following vector fields.

$$\vec{A} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

$$\vec{B} = y\,\hat{i} - x\,\hat{j}$$

$$\vec{C} = x^2\,\hat{j}$$