# Theoretical Physics
## Prof. Ruiz, UNC Asheville, doctorphys on YouTube
## Chapter J Notes. Spinors

**J1. Fun with Matrices.** Consider a general 2 x 2 matrix where each element can be complex in general. Note that we will be using notation common in physics.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

**1. The Trace**

$$Tr(A) = a + d$$

**2. The Transpose**

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Note that $Tr(A^T) = Tr(A)$ .

**3. The Complex Conjugate**

$$A^* = \begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix}$$

**Caution: In pure math $\overline{A}$ is used instead.**

**4. The Hermitian Conjugate**

$$A^\dagger = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} = (A^T)^* = (A^*)^T$$

**Caution: In pure math the star or H is used for the Hermitian conjugate.**

**5. The Determinant**

$$\det(A) \equiv |A| = ad - cb$$

## 6. The Inverse

$A^{-1}$ such that $AA^{-1} = I$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

**Caution: Some matrices do not have inverses.**

$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ does not have an inverse since

$BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ and you can never get $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

We will arrive at working out the inverse by trying a couple simple cases.

Let's see if we can find an inverse

$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for $A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$.

We note that

$\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ leads to

$3a = 1$, $3b = 0$, $4c = 0$, and $4d = 1$.

Solve these and we find

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$$

**Rule 1.** Swap the diagonal components and divide by the determinant.

**PJ1 (Practice Problem).** See if we are lucky and that our prescription works for complex numbers too. Try it for

$$A = \begin{bmatrix} 3+i & 0 \\ 0 & 4+2i \end{bmatrix}.$$

Let's now look for an inverse for

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}.$$

You see, we are trying to avoid solving four equations with four unknowns. Instead we want to psyche out the solution and in the process gain insight into how these inverses work.

$$\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ leads to}$$

$3a + c = 1$, $3b + d = 0$, $4c = 0$, and $4d = 1$.

Solving these, we are led to

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{12} \\ 0 & \frac{1}{4} \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 4 & -1 \\ 0 & 3 \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} 4 & -1 \\ 0 & 3 \end{bmatrix}$$

**Rule 2.** Put minus signs in front of the off-diagonal components.

**PJ2 (Practice Problem).** See if our prescription works for complex numbers too. Use the prescription to find the inverse for A and check to see if it works.

$$A = \begin{bmatrix} 3+i & 1+i \\ 0 & 4+2i \end{bmatrix}.$$

**PJ3 (Practice Problem).** To gain more confidence in our procedure, find the inverse to the following matrix using our prescription and check the result.

$$A = \begin{bmatrix} 3+i & 1+i \\ 2+3i & 4+2i \end{bmatrix}.$$

So we proudly summarize our procedure below.

Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \dfrac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ where $|A| = ad - cb$.

**J2. Unitary Groups.** First, we define the group of unitary n x n matrices where the binary operation is matrix multiplication. A matrix is unitary if

$$U^{\dagger} = U^{-1}$$

**1. U(1), i.e., 1 x 1 Matrices**

$$U = [u] \quad \text{and} \quad U^{\dagger} = [u^*]$$

$$UU^{\dagger} = [u][u^*] = [uu^*] = [1]$$

$$u(\theta) = e^{i\theta}$$

Closure: $u(\alpha)u(\beta) = e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)}$

Association: $[u(\alpha)u(\beta)]u(\gamma) = u(\alpha)[u(\beta)u(\gamma)]$

Identity: $I = u(0) = 1$

Inverse: $u^{-1}(\theta) = e^{-i\theta}$, consistent with $U^{-1} = U^{\dagger}$.

**2. SU(1)**

The "S" stands for "special." We must have the determinant equal to 1.

$$\det U = \det[u] = u = 1$$

So the group SU(1) contains the single matrix: $A = [1]$.

**3. SU(2)**

This group consists of the special unitary 2 x 2 matrices.

Start with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

As the determinant must be 1, we have

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

For the unitary matrices we must have

$$A^{\dagger} = A^{-1} \quad \text{which means} \quad A^{-1} = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix}.$$

The conditions are $a^* = d$ and $c = -b^*$ along with $|A| = ad - bc = 1$.

Let's write the real and imaginary components out, applying these conditions. Then,

$$A = \begin{bmatrix} a_r + ia_i & b_r + ib_i \\ -b_r + ib_i & a_r - ia_i \end{bmatrix}$$

for the general special unitary 2 x 2 matrix, where $a_r^2 + a_i^2 + b_r^2 + b_i^2 = 1$.

**"Derivation of the Pauli Matrices"**

**Wolfgang Pauli (1900-1958)**
Courtesy School of Mathematics and Statistics
University of St. Andrews, Scotland

Our general form the a matrix in the group SU(2).

$$A = \begin{bmatrix} a_r + ia_i & b_r + ib_i \\ -b_r + ib_i & a_r - ia_i \end{bmatrix}$$

Recall how you can write a vector in terms of basis unit vectors:

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \,.$$

Check out the same trick with matrices:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

But there is a natural expansion for our SU(2) matrices:

$$A = \begin{bmatrix} a_r + ia_i & b_r + ib_i \\ -b_r + ib_i & a_r - ia_i \end{bmatrix}$$

$$A = a_r\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + ia_i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b_r\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + ib_i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A = a_r\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + ia_i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + ib_r\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + ib_i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The Pauli matrices are $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

The anticommutator of A and B is defined as $\{A, B\} \equiv AB + BA$.

**Leopold Kronecker (1823-1891)**
Courtesy School of Mathematics and Statistics
University of St. Andrews, Scotland

The Kronecker Delta symbol is defined as

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

and named after the German mathematician Leopold Kronecker. It is a symmetric symbol.

**PJ4 (Practice Problem).** Show

$$\{\sigma_j, \sigma_k\} = 2\delta_{jk} I$$

The commutator of A and B is defined as $[A, B] \equiv AB - BA$.

**Tullio Levi-Civita (1873-1941)**
Courtesy School of Mathematics and Statistics
University of St. Andrews, Scotland

The Levi-Civita or permutation symbol is defined below. It is an antisymmetric symbol.

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3), (3,1,2) \text{ or } (2,3,1), \\ -1 & \text{if } (i,j,k) \text{ is } (1,3,2), (3,2,1) \text{ or } (2,1,3), \\ 0 & \text{if } i = j \text{ or } j = k \text{ or } k = i \end{cases}$$

**PJ5 (Practice Problem).** Show

$[\sigma_j, \sigma_k] = 2i\varepsilon_{jkl}\,\sigma_l$ and thus SU(2) is non-abelian.

Kronecker and Lev-Civita Definition Images Courtesy Wikipedia

**4. SO(2)**

This group consists of the special unitary orthogonal matrices. These are matrices where the columns and rows, thought of as vectors, are orthogonal. Remember or rotation matrix in 2D?

$$R(\theta)=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}.$$

$$A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

These matrices satisfy our orthogonal conditions since

$$a_{11}a_{12}+a_{21}a_{22}=\cos\theta\sin\theta-\cos\theta\sin\theta=0$$

$$a_{11}a_{21}+a_{12}a_{22}=-\cos\theta\sin\theta+\cos\theta\sin\theta=0$$

Note that now the transpose is the inverse.

$$A^{-1}=A^{T}$$

$$AA^{-1}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

**PJ6 (Practice Problem).** Show that the matrix

$$A=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$$

is an orthogonal matrix.

**PJ7 (Practice Problem).** Show that SO(2) is an abelian group.

**J3. Eigenvalues.** The spin of the electron has only two values, which we refer to as spin up or spin down. We represent these two states by a column vector having two elements. This two-valued vector is called a spinor.

$$\uparrow = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \downarrow = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The SU(2) matrices such as the Pauli matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

can serve as operators that change the state of a spinor. The first Pauli matrix flips each spinor.

$$\sigma_x \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \sigma_x \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let's try the second Pauli matrix.

$$\sigma_y \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \sigma_y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We expect to have the same physics for the $\sigma_x$ and $\sigma_y$ matrices. Comparing the pairs, we see the second one has an overall factor, which factor we call a phase factor.

$$\sigma_x \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \sigma_x \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \sigma_y \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^{i\frac{\pi}{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \sigma_y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e^{-i\frac{\pi}{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where we have expressed the phases using the Euler formula. Consider one of these.

$$\sigma_y \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^{i\frac{\pi}{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \text{ i.e., } c_1 = 0 \text{ and } c_2 = e^{i\frac{\pi}{2}}.$$

If we interpret the physics as $c_1{}^*c_1$ and $c_2{}^*c_2$, then the phase does not matter. It goes away. The value $c_1{}^*c_1$ in quantum mechanics gives the probability that we have spin up, while $c_2{}^*c_2$ gives the probability we have spin down.

A special result is obtained with the Pauli matrix labeled z.

$$\sigma_z \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \sigma_z \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We get the same state back with either a plus or minus sign. When you get the same thing back, we say you have an eigenvector for the operator and the value in front is called the eigenvalue.

If $A \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \lambda \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, then $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is an eigenvector of $A$ and $\lambda$ is the eigenvalue.

Let's find the eigenvectors and eigenvalues for the Paul matrix labeled x. We want to solve the following eigenvalue problem.

$$\sigma_x \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \lambda \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \text{, i.e., } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \lambda \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

The procedure is to first get everything on one side of the equation.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

Remember Cramer's formula from high school algebra? There, you want to solve two simultaneous linear equations.

$$ax + by = e \text{ and } cx + dy = f \text{, i.e., } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

You get

$$x = \frac{ed - bf}{ad - bc} = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \quad \text{and} \quad y = \frac{af - ce}{ad - bc} = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

Our eigenvalue problem has a vanishing numerator. So the denominator better vanish.

$$\det\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

Now it's time to find the eigenvectors that go with these. We go back to

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \lambda\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

and insert the eigenvalues.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = +1\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -1\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

For the first eigenvalue, we find

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$, giving $c_2 = c_1$.

We could choose each to be 1, but we should be sure the total probability is 1. So we go with

$$u = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The probability that you will find spin up is 1/2 and the probability that you will find spin down is 1/2 also. We say the vector is normalized when the total probability is 1 as it needs to be.

For the other eigenvector we obtain

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_2 \\ c_1 \end{bmatrix} = -\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \text{ giving } c_2 = -c_1 .$$

A satisfactory normalized eigenvector is

$$v = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Remember, if you multiply by an overall phase factor $e^{i\delta}$, the physics is still the same since you get the same equation relating the c-values and when finding c*c the phase factor goes away.

**PJ8 (Practice Problem).** Derive Cramer's rule for two simultaneous linear equations.

$$a_{11}x + a_{12}y = c_1 \text{ and } a_{21}x + a_{22}y = c_2 \text{, i.e., } \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Show

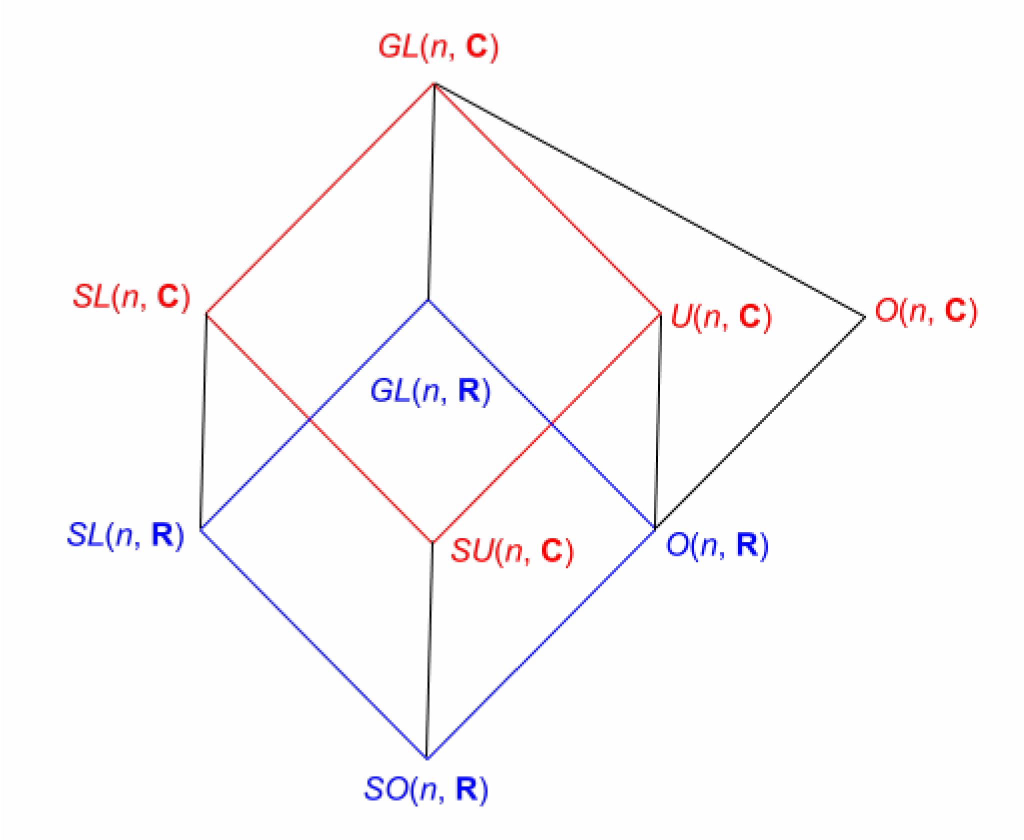
$$x = \frac{\begin{vmatrix} c_1 & a_{12} \\ c_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_{11} & c_1 \\ a_{21} & c_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} .$$

With three equations and three unknowns it gets more complicated. You deal with entities called cofactors and the procedure is more elaborate.

**PJ9 (Practice Problem).** Find the eigenvalues and eigenvectors for $\sigma_y$ .

## J4. Matrix Groups.

The top group is the general linear group of n x n matrices. These are your n x n matrices that have inverses.




It is understood that the binary operation for all the groups is your usual matrix multiplication.

The C stands for complex numbers and R indicates real for the various matrix groups. The top floor (red) uses the complex numbers. The groups emanating from *GL*(n, **C**) are subgroups. Can you identify the subgroups of the subgroups?