Theoretical Physics
Prof. Ruiz, UNC Asheville, doctorphys on YouTube
Chapter O Notes. Fourier Series

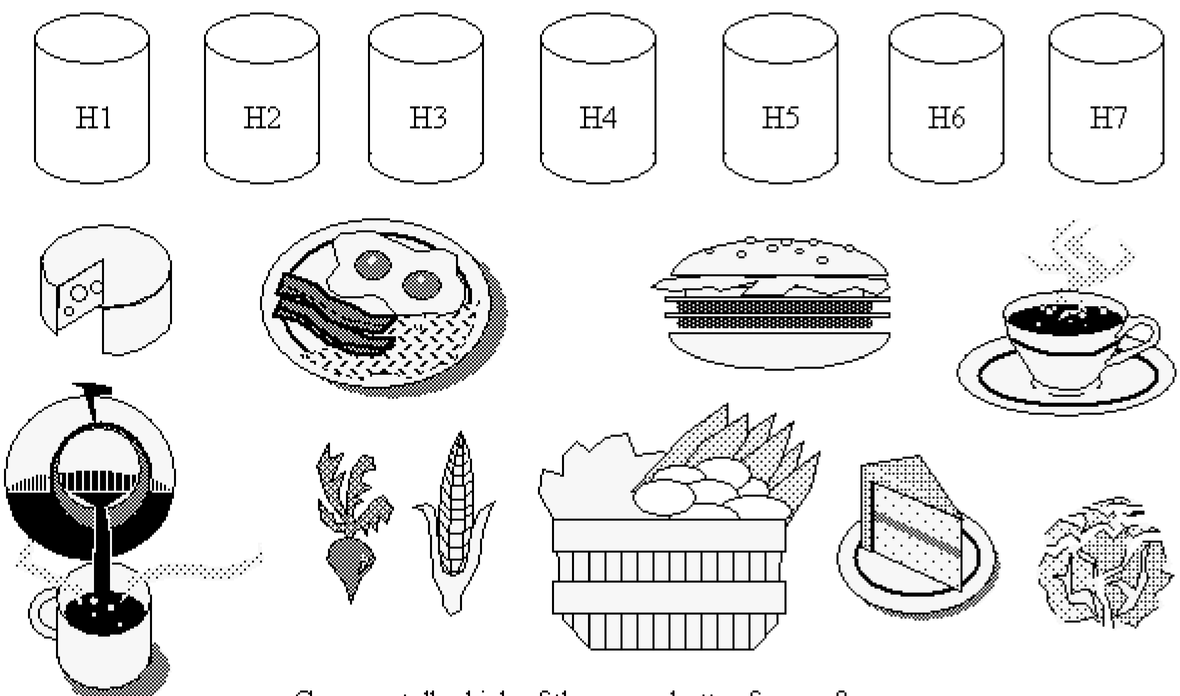
O1. Fourier's Theorem.

Jean Baptiste Joseph Fourier (1768-1830)
Courtesy School of Mathematics and Statistics
University of St. Andrews, Scotland

This section presents Fourier's Theorem suitable for the general student, where we concentrate on visualization. Later we will give the standard presentation for math and science majors.

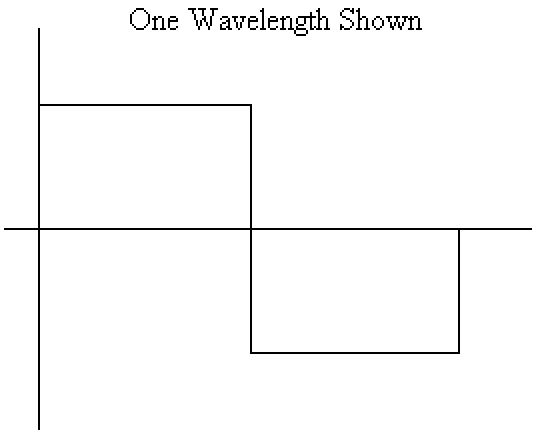
Fourier's Theorem
One can construct any periodic
wave having frequency f,
using sine waves with frequencies
f, 2f, 3f, 4f, ...
(the Harmonic Series).

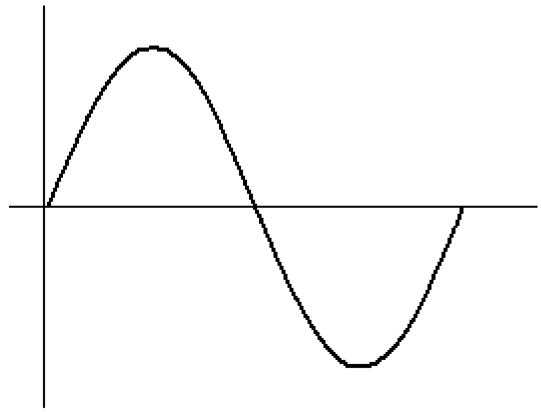
So the chef can cook up any period wave using the harmonic series as ingredients.



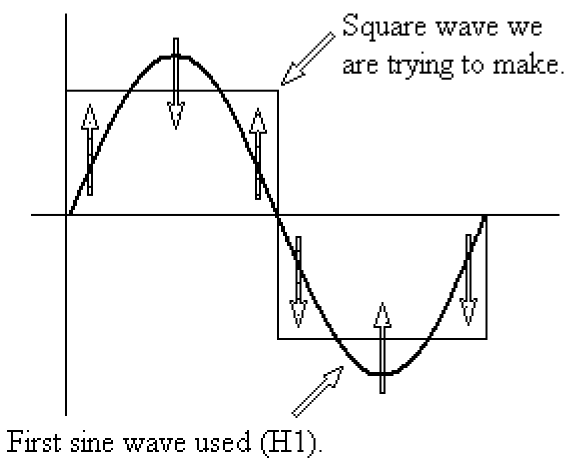
Can you tell which of these are better for you?

Let's order a square wave and watch the chef at work. The first ingredient is the first harmonic at "one full cup" which overshoots the crest a little. Among friends, is this close enough? Not really.



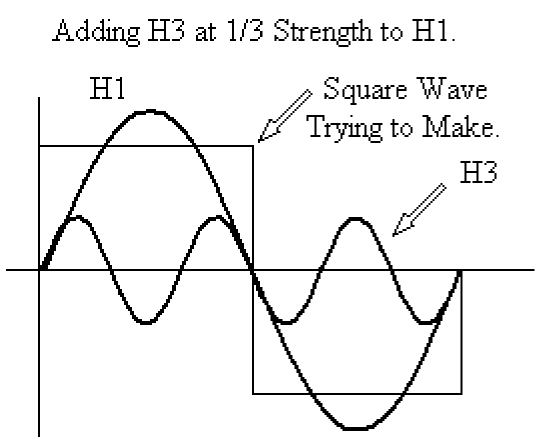


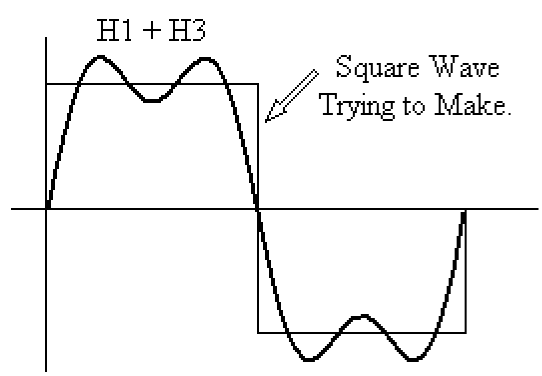
Below we analyze the corrections needed to improve our wave so far.



Note that the corrections are in step with a sine wave that has triple the frequency of our first harmonic. This means we need the third harmonic.

Here the chef instinctively throws in 1/3 cup of the third harmonic. Later, we will derive this result and all the other amplitude values. For now, we are focusing on visualization so that we can understand more fully what we will do later. Check out our H1+H3 sum.

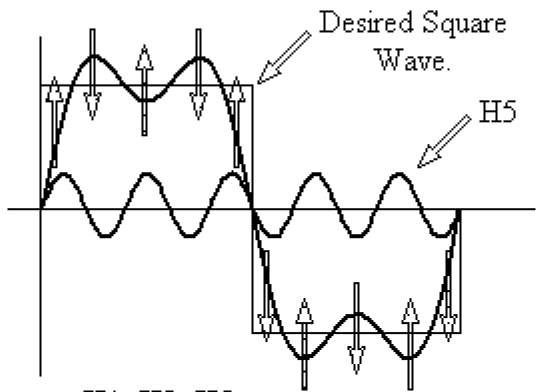


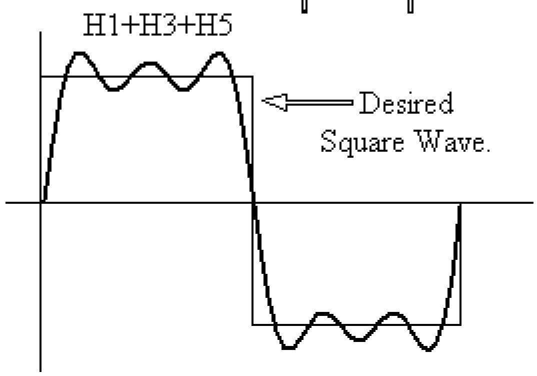


Remember that the harmonic series consists of a infinite series of sine waves where the n^{th} harmonic frequency is given by

$$f_n = nf_1 \text{, where } n = 1, 2, 3, ...$$

Note that the chef did not need to use the second harmonic, i.e., second ingredient from the cupboard. What is the next step?





We note the corrections needed to make our synthesized wave look even more like the desired goal, the square wave.

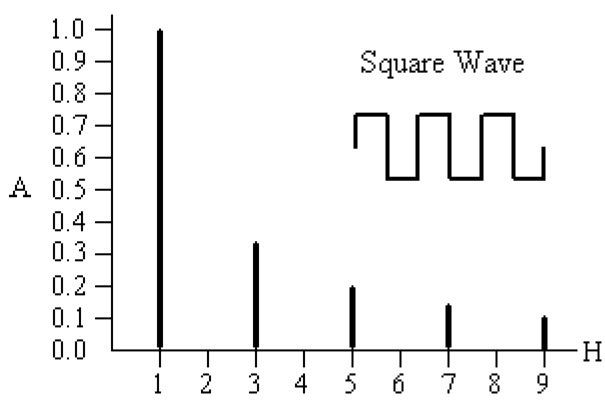
Note that the needed corrections are in step with the fifth harmonic.

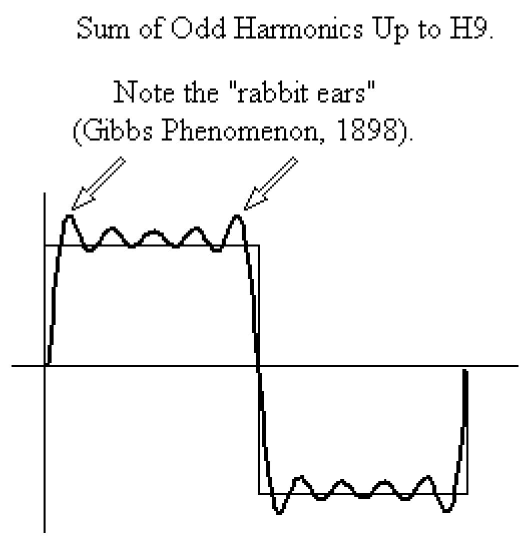
Here the chef instinctively knows to put a dash of the fifth harmonic at 1/5 cup. Again, we will derive this result later in this chapter.

Note that the chef did not need the fourth harmonic. Look at the wave so far at the left. The next correction will be in step with the seventh harmonic.

The recipe is to use the odd harmonics with amounts 1, 1/3, 1/5, 1/7, etc.

Our recipe can be conveniently written on an postcard to mail others as illustrated in the left figure below.

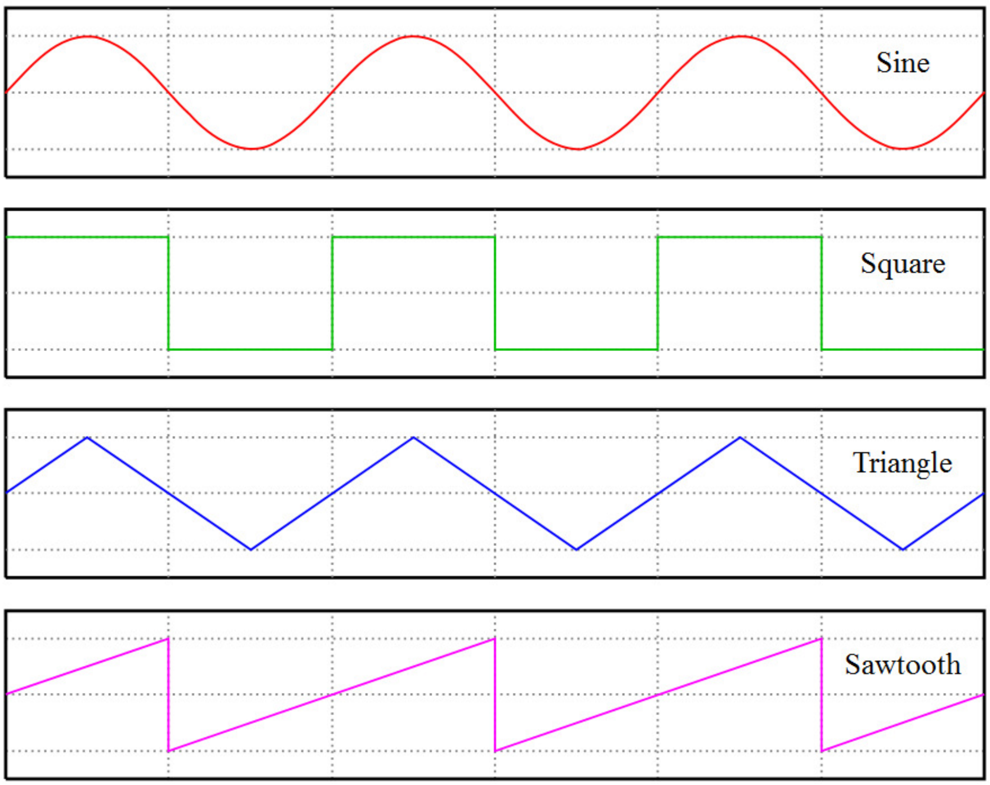




The above right figure shows a strange effect at the edges - the "rabbit ears." This is called the Gibbs phenomenon. As more and more odd harmonics are added, these "batman" ears close shut but overshoot the square wave's height. Mathematicians do not consider this a perfect match. But the areas of the square wave crest and synthesized wave match. This makes physicists happy enough. You can't hear a wave closed shut anyway because it has no time interval. So both the square wave and synthesized wave will sound exactly the same.

O2. Orthogonal Functions.

Since we are dealing here with periodic waves we will choose our wavelength λ for each wave to be 2π and center the wave so we can focus on x values from $-\pi$ to $+\pi$ for our analysis of one cycle.



Courtesy Omegatron, Wikimedia

PO1 (Practice Problem). Show how a periodic function $f(x)$ with wavelength 2π can be scaled to wavelength λ in the form $f(z)$. Express z in terms of x.

The mathematical form for adding our sine waves is

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(nx + \phi_n),$$

where we allow for the sine waves to be shifted in phase for the general case. The amplitudes A_n for our square-wave recipe are 1, 1/3, 1/5, ...

To allow for synthesizing a square wave that is shifted upward, we add a constant

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \sin(nx + \phi_n) .$$

Since $\sin(\alpha + \beta) = \cos\alpha \sin\beta + \sin\alpha \cos\beta$ from our first chapter, we can write

$$\sin(nx + \phi_n) = \cos(nx)\sin\phi_n + \sin(nx)\cos\phi_n$$

$$\sin(nx + \phi_n) = a_n \cos(nx) + b_n \sin(nx)$$

We can cash in the phases if we include the cosine series.

$$f(x) = A_0 + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

For some periodic wave, we would like to find the coefficients A_0, a_n, and b_n. Think of this challenge by an analogy with vectors. Suppose we know a vector $3\hat{i} + 4\hat{j}$ and want a formal way to pull out the components A_x, A_y, and A_z .

$$3\hat{i} + 4\hat{j} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

We can do this formally by taking dot products

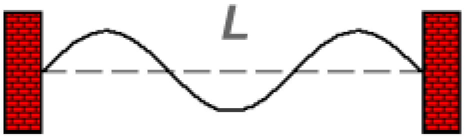
$\hat{i} \cdot (3\hat{i} + 4\hat{j}) = \hat{i} \cdot (A_x \hat{i} + A_y \hat{j} + A_z \hat{k})$, and similarly for $\hat{j}$ and $\hat{k}$,

finding

$3 = A_x$, $4 = A_y$, and $0 = A_z$.

Is there an analogous orthogonality relation for functions. We know from our eigenvalue problem that eigenspinors are orthogonal. What about the eigenfunctions for our square well?

Here is an eigenvalue problem we worked out: $H\psi_3 = E_3\psi_3$



$$3\frac{\lambda_3}{2} = L \qquad k_3 = \frac{2\pi}{\lambda_3}$$

$$k_3 = \frac{2\pi}{(2L/3)} = \frac{3\pi}{L} \qquad \psi_3(x) = A\sin(k_3 x) = A\sin(3\pi x/L)$$

The n^{th} eigenstate is

$$\psi_n(x) = A_n \sin(n\pi x/L) \text{ with energy } E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}.$$

Remember $\lambda_1 = 2L$. So we can write

$$\psi_n(x) = A_n \sin(2n\pi x/\lambda_1)$$

In keeping with our fundamental wavelength $\lambda = \lambda_1 = 2\pi$ for our periodic wave, then

$$\psi_n(x) = A_n \sin(nx)$$

The A_n is the normalization constant so that

$$\int_{-\pi}^{+\pi} \psi_n(x)^* \psi_n(x) dx = \int_{-\pi}^{+\pi} A_n^2 \sin^2(nx) dx = 1$$

We will do this integral using Euler's formula.

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\sin(nx) = \frac{e^{inx} - e^{-inx}}{2i}$$

$$\sin^2(nx) = \frac{e^{2inx} - 2 + e^{-2inx}}{-4}$$

$$\int_{-\pi}^{+\pi} A_n^2 \sin^2(nx)dx = 1$$

One of our integrals is this one.

$$\int_{-\pi}^{+\pi} e^{2inx} dx = \left. \frac{e^{2inx}}{2in} \right|_{-\pi}^{+\pi} = \frac{1}{2in}\left[\cos(2nx) + i\sin(2nx)\right]\Big|_{-\pi}^{+\pi}$$

$$= \frac{1}{2in}\left[\cos(2n\pi) + i\sin(2n\pi)\right] - \frac{1}{2in}\left[\cos(-2n\pi) + i\sin(-2n\pi)\right]$$

$$= \frac{1}{2in}\left[\cos(2n\pi) + i\sin(2n\pi)\right] - \frac{1}{2in}\left[\cos(2n\pi) - i\sin(2n\pi)\right]$$

$$= \frac{1}{2in}\left[\cos(2n\pi) - \cos(2n\pi)\right] - \frac{1}{2in}\left[\sin(2n\pi) - i\sin(2n\pi)\right]$$

$$= \frac{1}{2in}[0] - \frac{1}{2in}[0-0] = 0$$

PO2 (Practice Problem). Show $\int_{-\pi}^{+\pi} e^{-2inx} dx = 0$.

This leaves the integral of the middle term in

$$\sin^2(nx) = \frac{e^{2inx} - 2 + e^{-2inx}}{-4}$$

$$\int_{-\pi}^{+\pi} \sin^2(nx)dx = \int_{-\pi}^{+\pi} \frac{1}{2} dx = \left.\frac{x}{2}\right|_{-\pi}^{+\pi} = \frac{1}{2}\left[\pi - (-\pi)\right] = \pi$$

Using this result with our original integral

$$\int_{-\pi}^{+\pi} A_n^2 \sin^2(nx)dx = 1 ,$$

gives us the normalization constant

$$A_n = \frac{1}{\sqrt{\pi}} .$$

What about integrating two different eigenfunctions? This could be our dot product analogy. We consider integrals. Look at this one where n and m are now different.

$$\int_{-\pi}^{+\pi} \sin(nx)\sin(mx)dx = ?$$

$$\sin(nx)\sin(mx) = \left[\frac{e^{inx} - e^{-inx}}{2i}\right]\left[\frac{e^{imx} - e^{-imx}}{2i}\right]$$

Since $n \neq m$, all integrals have the form

$$\int_{-\pi}^{+\pi} e^{ipx} dx = \left.\frac{e^{ipx}}{ip}\right|_{-\pi}^{+\pi} = \frac{1}{ip}\left[\cos(px) + i\sin(px)\right]\Big|_{-\pi}^{+\pi} = 0$$

The functions are said to be orthogonal. The integral serves as the dot product.

Summary:

$$\int_{-\pi}^{+\pi} \sin(nx)\sin(mx) dx = \pi\delta_{nm}$$

$$\int_{-\pi}^{+\pi} \psi_n(x)^* \psi_m(x) dx = \delta_{nm}$$

O3. Fourier Series.

$$f(x) = A_0 + \sum_{m=1}^{\infty}\left[a_m \cos(mx) + b_m \sin(mx)\right]$$

You can use any summation variable you want in the above integral. Our aim is to project out one of the coefficients. Try this integral.

$$\int_{-\pi}^{+\pi} f(x)\sin(nx) dx$$

$$= \int_{-\pi}^{+\pi} A_0 \sin(nx) dx + \sum_{m=1}^{\infty} a_m \int_{-\pi}^{+\pi} \sin(nx)\cos(mx) dx$$

$$+ \sum_{m=1}^{\infty} b_m \int_{-\pi}^{+\pi} \sin(nx)\sin(mx) dx$$

$$= \int_{-\pi}^{+\pi} A_0 \sin(nx) dx + \sum_{m=1}^{\infty} a_m \int_{-\pi}^{+\pi} \sin(nx)\cos(mx) dx + \sum_{m=1}^{\infty} \pi b_m \delta_{nm}$$

$$= \int_{-\pi}^{+\pi} A_0 \sin(nx) dx + \sum_{m=1}^{\infty} a_m \int_{-\pi}^{+\pi} \sin(nx)\cos(mx) dx + \pi b_n$$

If the first two integrals are zero, we are in business.

PO3 (Practice Problem). Show that the first two integrals are zero. What about odd functions integrated over a symmetric region?

Summary:

$$f(x) = A_0 + \sum_{m=1}^{\infty} \left[a_m \cos(mx) + b_m \sin(mx) \right]$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin(nx) dx$$

PO4 (Practice Problem). Show the following.

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x) dx$$

PO5 (Practice Problem). Show the following using the exponential substitutions we did for the sine case.

$$a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos(nx) dx$$

To make all the constant integrals look similar, define a_0 as follows.

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x) dx = \frac{a_0}{2} = \frac{1}{2} \left[\frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) dx \right]$$

Then,

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left[a_m \cos(mx) + b_m \sin(mx) \right]$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin(nx) dx$$

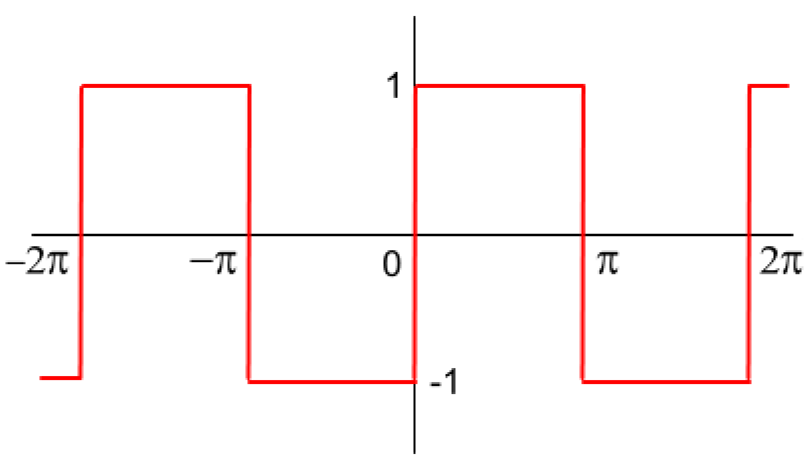
O4. The Square Wave.

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left[a_m \cos(mx) + b_m \sin(mx) \right]$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin(nx) dx$$



Since the above square wave is an odd function, the a_0 and a_n integrals are zero. The b_n integral is the one that will give nonzero values.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin(nx)dx = \frac{2}{\pi}\int_0^{\pi}\sin(nx)dx$$

$$b_n = -\frac{2}{\pi}\frac{\cos(nx)}{n}\bigg|_0^{\pi} = -\frac{2}{\pi}\frac{1}{n}\left[\cos(n\pi) - \cos(0)\right]$$

For even $n = 2k$, where $k = 1, 2, 3...$

$$b_{2k} = -\frac{2}{\pi}\frac{1}{2k}\left[\cos(2\pi k) - \cos(0)\right] = -\frac{2}{\pi}\frac{1}{2k}(1-1) = 0$$

For even $n = 2k - 1$, where $k = 1, 2, 3...$

$$b_{2k-1} = -\frac{2}{\pi}\frac{1}{2k-1}\left[\cos(2\pi k - \pi) - \cos(0)\right]$$

$$= -\frac{2}{\pi}\frac{1}{2k-1}(-1-1) = \frac{4}{\pi}\frac{1}{2k-1}$$

So we have our result from our visualization section.

$$b_n = \frac{4}{\pi}\frac{1}{n} \text{ for odd } n.$$

$$f(x) = \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x)...\right]$$