Theoretical Physics
Prof. Ruiz, UNC Asheville, doctorphys on YouTube
Chapter Q Notes. Laplace Transforms

Q1. The Laplace Transform.

Pierre-Simon Laplace (1749-1827)
Courtesy School of Mathematics and Statistics
University of St. Andrews, Scotland

The Laplace transform is defined as follows:

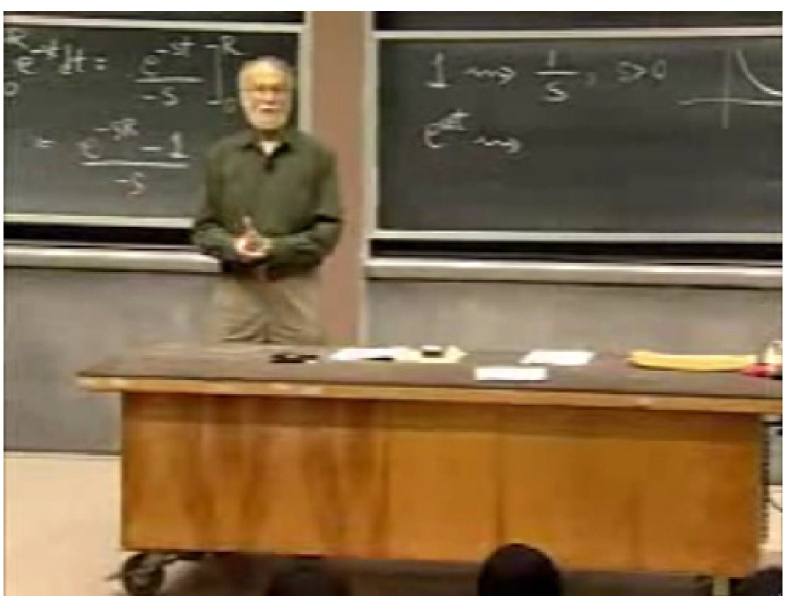
$$F(s) = \int_0^\infty f(t)e^{-st}dt \text{ , where } s > 0 .$$

This is also written with the notation shown below.

$$F(s) = L\{f(t)\}$$

In the spirit of our theoretical physics course, we would like to "derive" this formula.

We will follow Prof. Mattuck's "derivation," one that he gives in his Differential Equations course at MIT. At this point, do not be concerned with any applications of the Laplace transform. We will do that later.



Prof. Arthur Mattuck, MIT
Differential Equations

Prof. Mattuck uses the trick

$$x = e^{\ln x}$$

to show how to arrive at the Laplace transform from an infinite series.

We start with the power series

$$A(x) = \sum_{n=o}^{\infty} a_n x^n ,$$

where we write A(x) since the coefficients are given by the little a_n.

Since each n is one more than the previous one, $\Delta n = 1$ and we write

$$A(x) = \sum_{n=o}^{\infty} a_n x^n \Delta n \, .$$

Now to go to a continuous variable, we change this to an integral. Remember our three steps: 1)change the delta to "d", 2)rip off the n index and replace with a function of your promoted continuous variable, and 3)turn the summation sign into a "snake" (an integral sign).

$$A(x) = \int_0^{\infty} a(n) x^n dn$$

Everyone loves the natural base e, so we use the trick $x = e^{\ln x}$ and write the above as

$$A(e^{\ln x}) = \int_0^{\infty} a(n) \left[e^{\ln x} \right]^n dn$$

To increase our chances that this integral will not "blow up" in our face, we restrict $\ln x$ so that it is negative. Therefore, we want $\ln x \leq 0$. This occurs when $0 \leq x \leq 1$, visualized from the plot below.

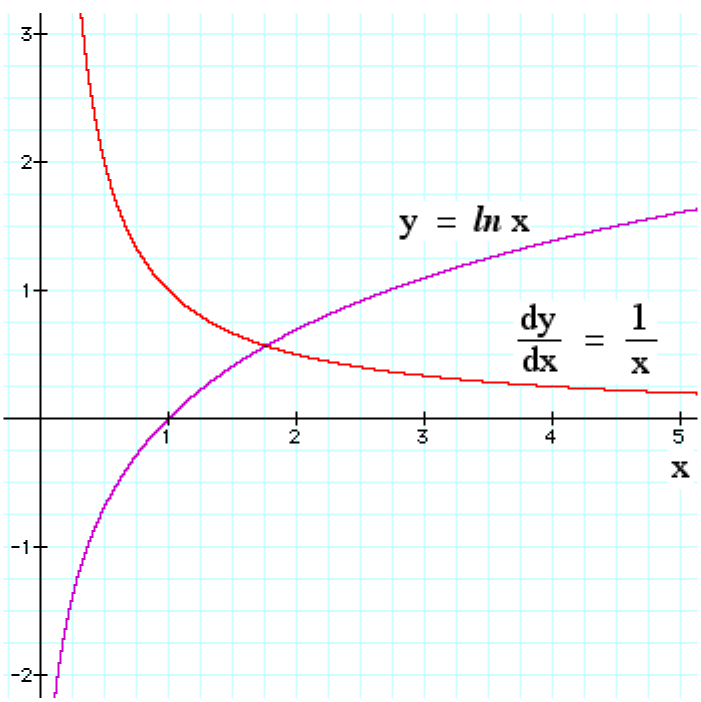


Image Courtesy The Australian Learning and Teaching Council, School of Physics, The University of New South Wales, Australia.

Therefore,

$\ln x = -s$, where $s > 0$.

We then have

$$A(e^{-s}) = \int_0^{\infty} a(n) \left[e^{-s} \right]^n dn \, ,$$

which is

$$A(e^{-s}) = \int_0^{\infty} a(n) e^{-sn} dn$$

Now two things look strange. First, that e^{-s} in the argument. So we fix this by writing

$$A(e^{-s}) \equiv F(s) = \int_0^\infty a(n) e^{-sn} dn$$

Next, that n looks strange since we typically reserve "n" for integers. So let's use the variable "t" instead. Then,

$$F(s) = \int_0^\infty a(t) e^{-st} dt$$

Finally, since we originally started with "A" matched to the little "a," let's replace a(t) with f(t) so that "F" is matched with little "f."

$$\boxed{F(s) = \int_0^\infty f(t) e^{-st} dt} \quad \text{where} \quad s > 0 .$$

This is the Laplace transform.

Q2. Evaluating Laplace Transforms.

Still, do not concern yourself with any applications. Let's become familiar with the Laplace transform by evaluating Laplace transforms for some common functions f(t).

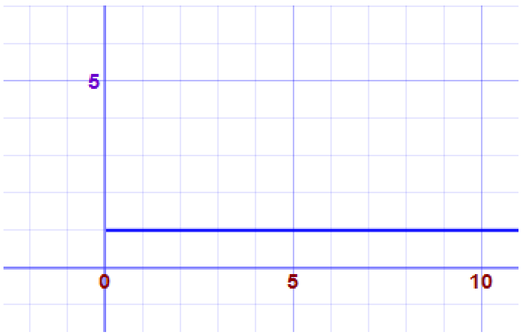
The function $f(t) = 1$

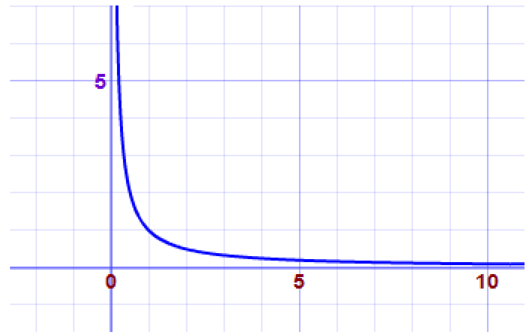
$$F(s) = \int_0^\infty f(t) e^{-st} dt = \int_0^\infty e^{-st} dt$$

$$F(s) = \left. \frac{e^{-st}}{-s} \right|_0^\infty = 0 - \left[\frac{1}{-s} \right]$$

$$F(s) = \frac{1}{s}, \text{ where } s > 0 .$$

$$f(t) = 1 \qquad\qquad F(s) = 1/s$$





Images made at www.mathisfun.com, Function Grapher and Calculator.

Think of these two graphs as residing in two separate spaces, like two worlds. The simple constant function in t-space appears as 1/s in s-space. See the analogy below where we transform from the real night sky to Van Gogh's *Starry Night* (1889).

Normal Space

Van Gogh Space

The function $f(t) = e^{at}$

$$F(s) = \int_0^\infty e^{at} e^{-st} dt = \int_0^\infty e^{(a-s)t} dt$$. We must have $s > a$.

$$F(s) = \left. \frac{e^{(a-s)t}}{a-s} \right|_0^\infty = 0 - \left[\frac{1}{a-s} \right] = \frac{1}{s-a} \quad \text{for} \quad s > a$$

Summary: $F(s)=\dfrac{1}{s-a}$, where $s>a$.

The function $g(t)=e^{at}f(t)$ and the Laplace Transform Shifting Property

$$G(s)=\int_0^\infty e^{at}f(t)e^{-st}dt=\int_0^\infty f(t)e^{-(s-a)t}dt$$

$G(s)=F(s-a)$, where $s>a$.

PQ1 (Practice Problem). Find the Laplace transform for $f(t)=e^{at}$ by using the shifting property $G(s)=F(s-a)$, where $g(t)=e^{at}f(t)$ with $f(t)=1$.

The functions $f(t)=\cos\omega t$ and $\sin\omega t$

We use the Real-Imaginary Trick and take $f(t)=\cos\omega t+i\sin\omega t=e^{i\omega t}$. The imaginary number "i" keeps the two the functions separate for us.

Use $L\{e^{at}\}=\dfrac{1}{s-a}$ with $a=i\omega$.

$$L\{e^{at}\}=\frac{1}{s-i\omega}$$

$$L\{e^{at}\}=\frac{1}{s-i\omega}\frac{s+i\omega}{s+i\omega}=\frac{s+i\omega}{s^2+\omega^2}$$

$$L\{\cos\omega t\}=\frac{s}{s^2+\omega^2}\quad\text{and}\quad L\{\sin\omega t\}=\frac{\omega}{s^2+\omega^2}$$

The cosine and sine functions can also been worked with in the traditional manner.

The function $f(t) = \cos \omega t$ **(the traditional way).**

Let's use the backward Euler formula: $\cos \omega t = \dfrac{e^{i\omega t} + e^{-i\omega t}}{2}$

$$F(s) = \frac{1}{2}\int_0^\infty e^{i\omega t} e^{-st} dt + \frac{1}{2}\int_0^\infty e^{-i\omega t} e^{-st} dt$$

Now use our former result $L\{e^{at}\} = \dfrac{1}{s-a}$ where $s > a$.

$$F(s) = \frac{1}{2}\left[\frac{1}{s - i\omega} + \frac{1}{s + i\omega}\right]$$

$$F(s) = \frac{1}{2}\left[\frac{2s}{s^2 + \omega^2}\right]$$

$$F(s) = \frac{s}{s^2 + \omega^2}$$

PQ2 (Practice Problem). Use the traditional way to show that the Laplace transform for $f(t) = \sin \omega t$ is

$$F(s) = \frac{\omega}{s^2 + \omega^2}.$$

PQ3 (Practice Problem). Use the shifting property that the Laplace transform of $g(t) = e^{at} f(t)$ is $G(s) = F(s-a)$ to show that the Laplace transform of $f(t) = e^{-at}\cos\omega t$ is

$$F(s) = \frac{s+a}{(s+a)^2 + \omega^2}.$$

PQ4 (Practice Problem). Use the shifting property to show that the Laplace transform of $f(t) = e^{-at}\sin\omega t$ is

$$F(s) = \frac{\omega}{(s+a)^2 + \omega^2}.$$

The function $f(t) = t^n$

$$F(s) = \int_0^\infty t^n e^{-st}\,dt$$

$$F(s) = \left[-\frac{d}{ds}\right]^n \int_0^\infty e^{-st}\,dt$$

$$F(s) = \left[-\frac{d}{ds}\right]^n L\{1\}$$

$$F(s) = \left[-\frac{d}{ds}\right]^n \frac{1}{s}$$

$$F(s) = \frac{n!}{s^{n+1}}$$

By the way, note also that the Laplace transform is a linear operation

$$L\{\alpha f(t)+\beta g(t)\}=\alpha L\{f(t)\}+\beta L\{g(t)\}$$

PQ5 (Practice Problem). Explain or show why $L\{0\}=0$.

Our Laplace Transform Table $(s>a>0)$.

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$

Q3. The Laplace Transform of a Derivative.

$$L\{f'(t)\} = \int_0^\infty \frac{df}{dt} e^{-st} dt$$

We will use integration by parts. Always think of integration by parts as being related to the product rule for differentiation.

$$\frac{d}{dt}\left[f e^{-st} \right] = \frac{df}{dt} e^{-st} - fse^{-st}$$

Therefore,

$$L\{f'(t)\} = \int_0^\infty \frac{d}{dt}\left[f e^{-st} \right] dt + \int_0^\infty s f e^{-st} dt$$

$$L\{f'(t)\} = fe^{-st}\Big|_0^\infty + sL\{f(t)\}$$

It is important that $f(t) < e^{st}$ so the integral converges.

$$L\{f'(t)\} = fe^{-st}\Big|_0^\infty + sL\{f(t)\} = 0 - f(0) + sF(s)$$

$$\boxed{L\{f'(t)\} = sF(s) - f(0)}$$

Next, consider a Laplace transform of a second derivative. The trick is to consider the second derivative as a first derivative of something.

$$L\{f''(t)\} = L\{g'(t)\} \quad \text{with} \quad g(t) = f'(t)$$

$$L\{g'(t)\} = sG(s) - g(0) \quad \text{with} \quad G(s) = sF(s) - f(0)$$

$$\boxed{L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)}$$

Q4. Differential Equations: Radioactive Decay. Now comes the application!

1. Take the Laplace Transform of Your Differential Equation

The Differential Equation MELTS into an Algebraic Equation in s-Space

Normal Space	**Van Gogh Space**
	

2. Solve the Algebraic Equation in s-Space

3. Use Your Laplace Transform Table to Come Home to Regular Space

Your "Laplace Transform Table" is your porthole to return home.

Radioactive Decay

The infinitesimal change in the number of radioactive decay particles is proportional to the product of the number of particles remaining and the infinitesimal time interval. The minus sign indicates a decrease in radioactive particles remaining.

$$dN = -\lambda N dt \text{ with } \lambda > 0$$

Therefore, the rate of change is proportional to what you have left.

$$\frac{dN}{dt} = -\lambda N$$

PQ6 (Practice Problem). Solve the above differential equation using standard methods. We will solve this differential equation using Laplace transforms.

We will use this form for our equation.

$$\frac{dN(t)}{dt} + \lambda N(t) = 0$$

1. Take the Laplace Transform

$$L\{\frac{dN(t)}{dt}\} + L\{\lambda N(t)\} = L\{0\}$$

Remember that $L\{f'(t)\} = sF(s) - f(0)$. Therefore.

$$sF(s) - N(0) + \lambda F(s) = 0$$

2. Solve Your Algebraic equation

$$sF(s) + \lambda F(s) = N(0)$$

$$F(s)(s+\lambda) = N(0)$$

$$F(s) = \frac{N(0)}{s+\lambda}$$

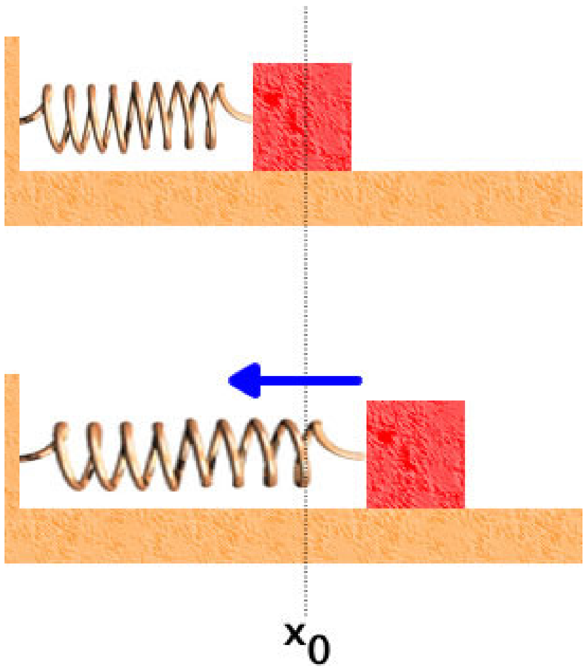
3. Use the Laplace Transform Table to Get Your Solution

$$N(t) = L^{-1}\{F(s)\} = N(0)\,L^{-1}\{\frac{1}{s+\lambda}\}$$

e^{at}	$\frac{1}{s-a}$

$$N(t) = N(0)e^{-\lambda t}$$

Q5. The Damped Harmonic Oscillator.



A Mass is Attached to Spring.
Courtesy David M. Harrison
Department of Physics, University of Toronto

Recall our mass on the spring.

$$F = -kx - bv = ma$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

Let's solve this with initial conditions $x(0) = A$ and $\frac{dx(0)}{dt} \equiv v(0) = -\frac{b}{2m}A$.

We will need the Laplace transform for the first and second derivatives:

$$L\{f'(t)\} = sF(s) - f(0) \text{ and } L\{f''(t)\} = s^2F(s) - sf(0) - f'(0).$$

1. Take the Laplace Transform

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

$$m\left[s^2F(s) - sx(0) - v(0)\right] + b\left[sF(s) - x(0)\right] + kF(s) = 0$$

With our initial conditions $x(0) = A$ and $v(0) = -\frac{b}{2m}A$ we have

$$m\left[s^2F(s) - sA + \frac{b}{2m}A\right] + b\left[sF(s) - A\right] + kF(s) = 0$$

2. Solve Your Algebraic equation

$$F(s)\left[ms^2 + bs + k\right] = Ams - A\frac{b}{2} + Ab$$

$$F(s) = A\frac{(ms + \frac{b}{2})}{ms^2 + bs + k}$$

3. Use the Laplace Transform Table to Get Your Solution

$$x(t) = L^{-1}\{F(s)\} = L^{-1}\{A\frac{(ms + \frac{b}{2})}{ms^2 + bs + k}\}$$

From our table, it appears we have something close to this one below. This looks promising so complete the square in the denominator.

$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$

First divide by the mass m.

$$F(s) = A\frac{(s + \frac{b}{2m})}{s^2 + \frac{b}{m}s + \frac{k}{m}}$$

$$F(s) = A\frac{(s + \frac{b}{2m})}{\left[s + \frac{b}{2m}\right]^2 + \frac{k}{m} - \left[\frac{b}{2m}\right]^2}$$

Matching this with $\dfrac{s+a}{(s+a)^2+\omega^2}$ gives us

$$a=\frac{b}{2m} \quad \text{and} \quad \omega^2=\frac{k}{m}-\left[\frac{b}{2m}\right]^2$$ with the solution

$$x(t)=Ae^{-at}\cos\omega t$$

Note that if there is no friction, i.e., $b=0$, then $\omega^2=\dfrac{k}{m}$. This angular frequency for the frictionless oscillator is defined as $\omega_0^2=\dfrac{k}{m}$, i.e., $\omega_0=\sqrt{\dfrac{k}{m}}$. In classical mechanics one often defines $\beta=\dfrac{b}{2m}$, calling this the damping coefficient.

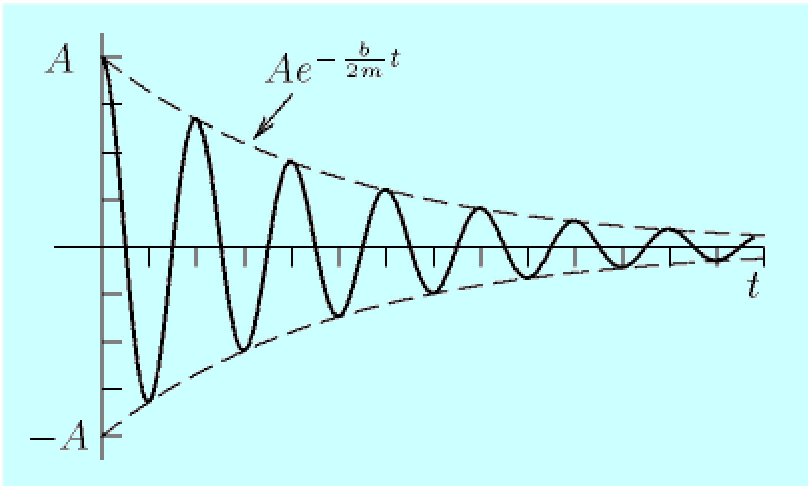
With these definitions:

$$\omega_0=\sqrt{\frac{k}{m}} \quad \text{and} \quad \beta=\frac{b}{2m},$$

$$a=\frac{b}{2m}=\beta \quad \text{and} \quad \omega^2=\frac{k}{m}-\left[\frac{b}{2m}\right]^2=\omega_0^2-\beta^2.$$

Summary: For the initial conditions $x(0) = A$ and $v(0) = -\frac{b}{2m}$,

$$x(t) = Ae^{-\beta t} \cos \omega t \text{, where } \beta = \frac{b}{2m} \text{ and } \omega^2 = \omega_0^2 - \beta^2.$$



Courtesy User LP, Wikimedia Commons

Since $\omega = 2\pi f = \frac{2\pi}{T}$, the period of the damped oscillations is $T = \frac{2\pi}{\omega}$, i.e.,

$$T = \frac{2\pi}{\sqrt{\omega_0^2 - \beta^2}}.$$