Theoretical Physics
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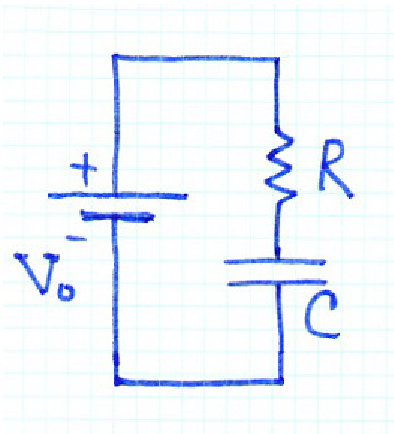
Chapter R Notes. Convolution

R1. Review of the RC Circuit. The convolution is a "difficult" concept to grasp. So we will begin this chapter with a review of the basic RC circuit, which we plan to use for our discussing convolution.



The voltage from bottom to top on the left side of the circuit is V_0, which must be the same if you go up the right side:

$$V_0 = V_R + V_C .$$

The voltage across the resistor is given by Ohm's Law:

$$V_R = IR$$

This law states that if you increase the voltage across a resistor, you increase the current. Think of a simple circuit with a battery and resistor. The greater the voltage, the greater the current. If you replace the resistor with one with a greater resistance, then you decrease the current. Ohm's Law specifically states the resistance for a given resistor is constant. Then you have a linear graph when you plot the voltage against the current.

For the capacitor, the greater the voltage V_C, the greater the charge q in a linear fashion. So we write the voltage V_C as proportional to the charge q.

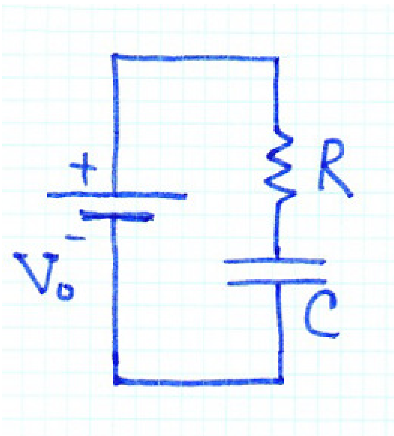
$$V_C \sim q$$

The capacitance C is a constant that gives us a measure of how easily the capacitor can store lots of charge. If the capacitance is greater, then it can store more charge q, given a fixed voltage V_C. To make this come out right, we divide the charge by the capacitance:

$$V_C = \frac{q}{C}$$

Both $V_R = IR$ and $V_C = \dfrac{q}{C}$ have their limits. If you zap either the resistor or the capacitor with too high a voltage, you can waste them and thus burn them out.

With these substitutions, our equation $V_0 = V_R + V_C$ becomes



$$V_0 = IR + \frac{q}{C}.$$

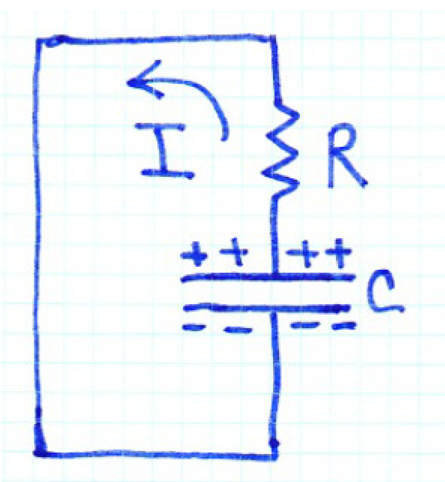
Imagine attaching the battery at time $t = 0$ where there is no charge initially on the capacitor. Then, initially there is a rush of current where

$$V_R(0) = V_0 = IR \quad \text{and} \quad V_C(0) = 0 \quad \text{since}$$

$$q(0) = 0.$$

The capacitor is being charged up. After charging, i.e., waiting a long time, we have no more current.

$$V_R(\infty) = 0 \quad \text{and} \quad V_C(\infty) = \frac{q(\infty)}{C} \quad \text{with } q(\infty) = CV_0$$



Let's remove the battery by making $V_0 = 0$. Assume there is some initial charge $q(0) = CV_0$ stored on the capacitor due to prior charging. Now we have our standard discharge situation. Note that

$$I = \frac{dq}{dt},$$

i.e., the current is the flow of charge per unit time interval. The differential equation describing the discharge is

$$0 = R\frac{dq}{dt} + \frac{q}{C}.$$

We can solve this equation by separating the variables q and t.

$$R\frac{dq}{dt} = -\frac{q}{C}$$

$$\frac{1}{q}dq = -\frac{1}{RC}dt$$

$$\int_{q(0)}^{q(t)}\frac{1}{q}dq = -\frac{1}{RC}\int_0^t dt$$

$$\ln q\Big|_{q(0)}^{q(t)} = -\frac{1}{RC}t\Big|_0^t$$

$$\ln q(t) - \ln q(0) = -\frac{t}{RC}$$

$$\ln\frac{q(t)}{q(0)} = -\frac{t}{RC}$$

$$\frac{q(t)}{q(0)} = e^{-\frac{t}{RC}}$$

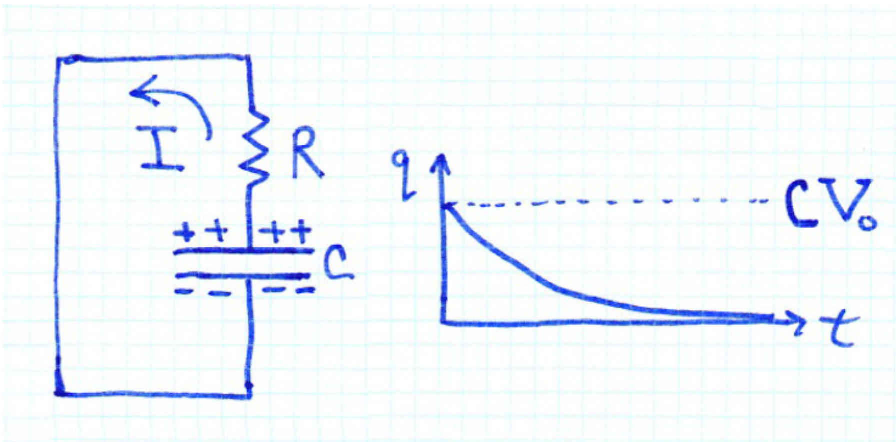
$$q(t) = q(0)e^{-\frac{t}{RC}}$$

With our prior charge of CV_0, we have.

$$q(t) = CV_0e^{-\frac{t}{RC}}$$

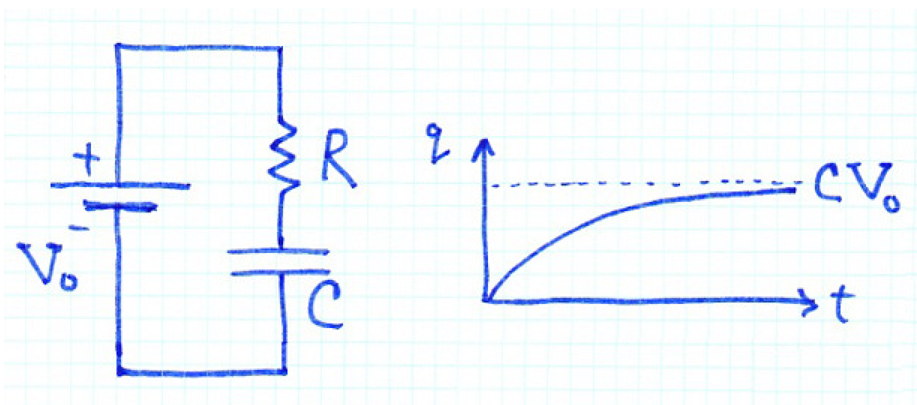
Summary of the Discharging Circuit.

$$q(t) = CV_0 e^{-\frac{t}{RC}}$$

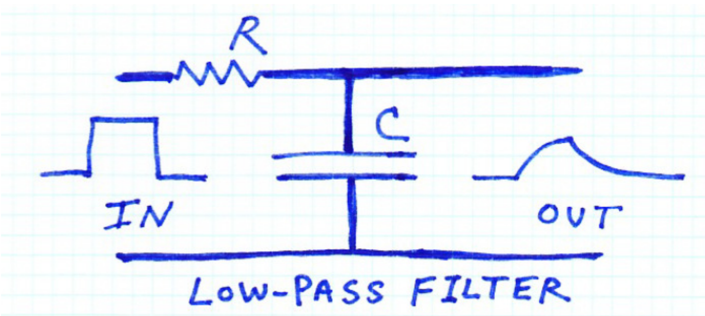


PR1 (Practice Problem). Solve the differential equation for the charging circuit and show that

$$q(t) = CV_0 (1 - e^{-\frac{t}{RC}}) .$$



R2. Square Pulse through Low-Pass Filter. We consider our RC circuit oriented to receive an input wave. We apply a pulse voltage. The capacitor will begin to charge up and then discharge. If the pulse-time is short enough, the capacitor will not fully charge. When the pulse voltage drops to zero, the capacitor will discharge.



The output shown is the voltage across the capacitor. This RC circuit is a low-pass filter by the way.

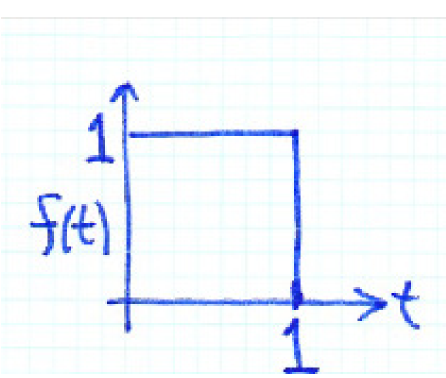
The physics can be described by a convolution. You will understand convolution much better with this approach because you will know everything about this circuit here through conventional methods. We are going to simply cast this in terms of convolution.

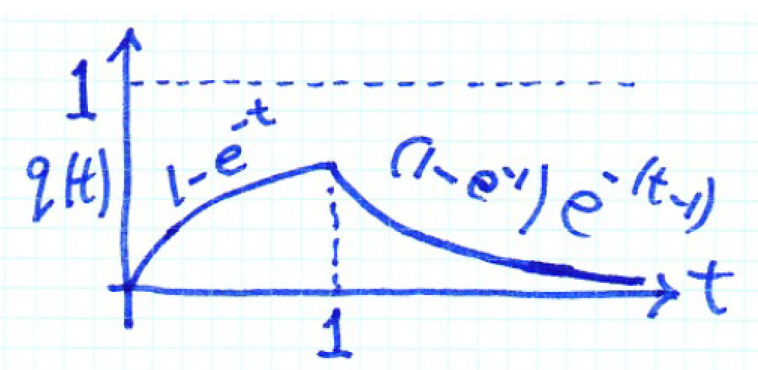
Even with this said, convolution will still be "difficult" so we make the following assignments in order to concentrate on the pure math: $R=1$ and $C=1$. In other words, we have a 1-ohm resistor and a 1-Farad capacitor. We also take $V_0=1$, i.e., our pulse voltage is 1 volt. And you probably guessed that we will apply the pulse for 1 second so that each parameter is 1. Then the charging and discharging equations simplify.

Charging: $q(t)=CV_0(1-e^{-\frac{t}{RC}})$ becomes $q(t)=1-e^{-t}$

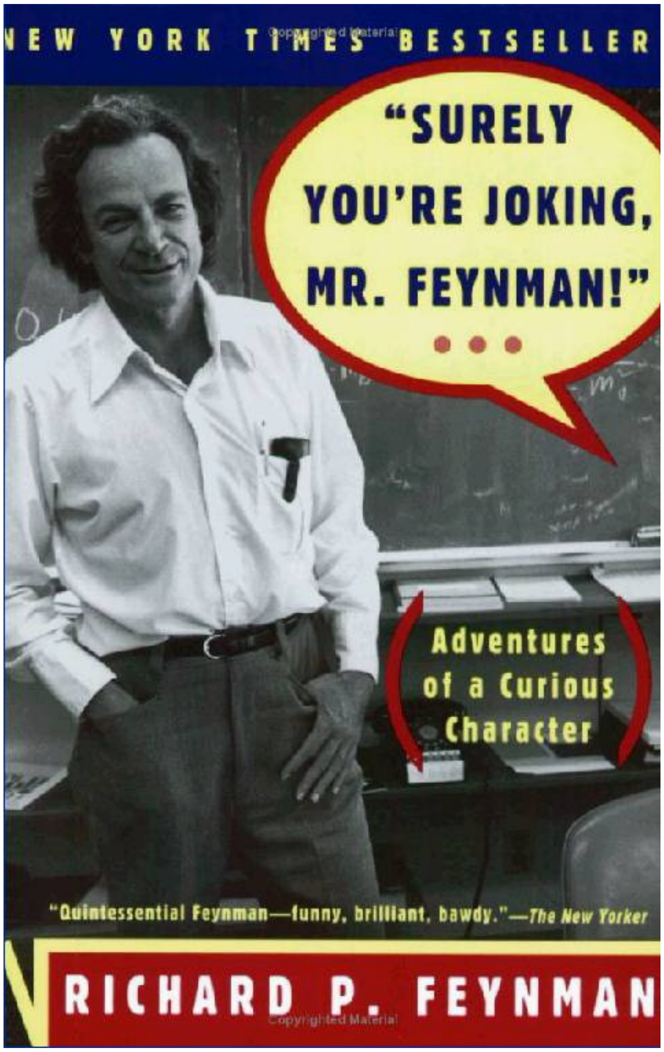
Discharging $q(t)=CV_0 e^{-\frac{t}{RC}}$ becomes $q(t)=e^{-t}$.

Our applied voltage is represented by $f(t)$. The capacitor starts charging. But the applied voltage drops to zero after 1 second. At the 1-second point the capacitor has charge $1-e^{-1}$. From this moment on the discharge kicks in. Note that the discharge is referenced to time $t-1$, time greater than 1 second. The functions match at $t=1$.





R3. Convolution.



You know that an inspiration of our course is Richard Feynman, an outstanding theoretical physicist.

Feynman always encouraged others to work things out for themselves in order to really understand what is going on. As early as a teenager, Feynman kept notebooks in which he worked out details for himself in his own way.

Once Feynman commented along these lines in reference to experimentalists.

"I suddenly realized why Princeton was getting results. They were working with the instrument. They *built* the instrument; they knew where everything was, they knew how everything worked, ... It was wonderful! Because they *worked* with it. They didn't have to sit in another room and push buttons!" Richard Feynman.

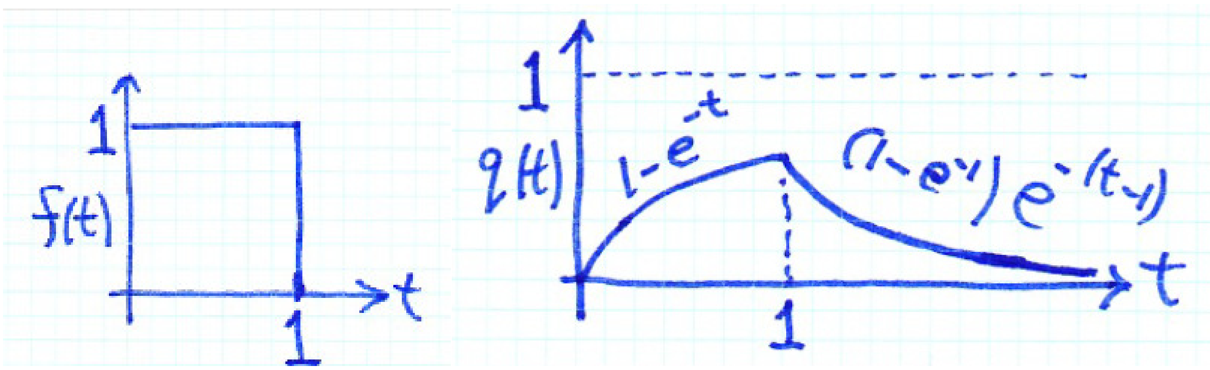
Source: *Surely you're joking Mr. Feynman! (Adventures of a Curious Character*, by Richard P. Feynman (Author), Ralph Leighton (Author), and Edward Hutchings (Editor), and Albert R. Hibbs (Introduction), published by W. W. Norton & Company (April 17, 2997). Book availabe at www.amazon.com.

By 1985, the year when this book was first published, many so-called "Feynman" stories had amassed in the folklore of physics over the years. Many of these are found in this book and still more in the companion volume *What Do You Care What Other People Think?: Further Adventures of a Curious Character*.

Following Feynman's stress on the importance of working out details and building the instrument, we have prepared the way for convolution this way. We are going to use the example we "built" in the previous section - the basic RC circuit covered in the introductory physics course, but applied with a pulse voltage.

We sacrifice formal mathematical derivation (the room with the buttons) for insight.

You have already met your first convolution. The function at the right below is a convolution of a square pulse $f(t)$ with the discharging function $g(t) = e^{-t}$ of the capacitor. We will show you what we mean by this now.



Start with $q(t)$ where $t \geq 1$. Let's play with this solution.

$$q(t) = (1 - e^{-1})e^{-(t-1)}$$

$$q(t) = e^{-(t-1)} - e^{-t}$$

$$q(t) = e^{-t}(e - 1)$$

$$q(t) = e^{-t}(e^{1} - e^{0})$$

$$q(t) = e^{-t}\, e^{u}\Big|_{0}^{1} = e^{-t}\int_{0}^{1} e^{u}\,du$$

What about for $t \leq 1$. Let's check.

$$q(t) = e^{-t}\int_{0}^{t} e^{u}\,du = e^{-t}\, e^{u}\Big|_{0}^{t} = e^{-t}(e^{t} - 1) = 1 - e^{-t}$$

Since $f(u) = 1$ for $u \leq 1$ and 0 elsewhere, we can write for all t:

$$q(t) = e^{-t}\int_{0}^{t} f(u)e^{u}\,du \,.$$

Our equation $q(t) = e^{-t}\int_0^t f(u)e^u du$ can also be written as

$$q(t) = \int_0^t f(u)e^{-(t-u)}du \ .$$

This is your convolution. So what do we mean mathematically when we state that the function $q(t)$ is a convolution of a square pulse $f(t)$ with the discharging function $g(t) = e^{-t}$?

We simply mean this "convoluted" integral.

$$q(t) = \int_0^t f(u)e^{-(t-u)}du$$

or more formally the following.

$$q(t) = \int_0^t f(u)g(t-u)du$$

We convolute f with g. Or we take the convolution of f and g. The notation for convolution is given below.

$$f(t) * g(t) = \int_0^t f(u)g(t-u)du$$

R4. Convolution is Commutative. We demonstrate in this section, using our example, that the convolution operator is commutative.

Our two functions for $0 \le t$: $f(t) = 1$ for $0 \le t \le 1$ and $g(t) = e^{-t}$.

The convolution

$$f(t) * g(t) = \int_0^t f(u) g(t-u) du$$

in the last section gave us

$$f(t) * g(t) = \int_0^t f(u) e^{-(t-u)} du = e^{-t} \int_0^t f(u) e^{u} du$$

What about

$$g(t) * f(t) = \int_0^t g(u) f(t-u) du \text{ ?}$$

How do you shift the pulse function? It is just 1 for the interval or 1 second. This is best handled by a change of variable. Define a new integration variable

$z = t - u$. Then $du = -dz$ and

as u goes from 0 to t, we have z going from t to 0.

Our convolution $g(t) * f(t) = \int_0^t e^{-u} f(t-u) du$ with the new variable is

$$g(t) * f(t) = \int_t^0 e^{z-t} f(z)(-dz) = \int_0^t e^{z-t} f(z) dz$$

In the last step the minus was used to flip the order of the integration limits.

$$g(t) * f(t) = e^{-t} \int_0^t e^{z} f(z) dz$$

But this is $f(t) * g(t) = e^{-t} \int_0^t f(u) e^{u} du$ since we can choose any letter for the integration variable.

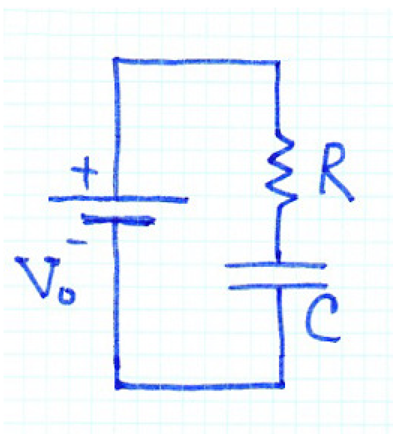
Therefore, convolution is commutative.

$$f(t) * g(t) = g(t) * f(t)$$

$$f(t) * g(t) = \int_0^t f(u)g(t-u)du$$

$$g(t) * f(t) = \int_0^t g(u)f(t-u)du$$

R5. The Laplace Transform and Convolution. We return to our circuit problem to see how Laplace transforms are related to convolutions.



We return to the RC circuit.

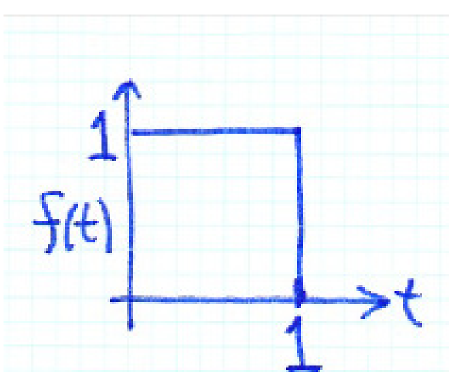
$$V_0 = IR + \frac{q}{C}$$

With $V_0 = f(t)$, $R = 1$, and $C = 1$,

$$f(t) = I + q \text{, i.e., } \frac{dq}{dt} + q = f(t)$$

We take the Laplace transform of both sides.

$$L\{\frac{dq}{dt}\} + L\{q\} = L\{f(t)\}$$



We need the Laplace transform of our box function.

$$L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

But we will not need to actually calculate it since we are aiming for a more general relationship.

$$L\{\frac{dq}{dt}\} + L\{q\} = L\{f(t)\}$$

$$sQ(s) - q(0) + Q(s) = F(s)$$

There is no initial charge, therefore $q(0) = 0$.

$$sQ(s) + Q(s) = F(s)$$

$$Q(s)(s+1) = F(s)$$

$$Q(s) = F(s)\frac{1}{s+1}$$

But we know the answer is

$$q(t) = f(t) * g(t) = \int_0^t f(u)g(t-u)du$$, where $g(t) = e^{-t}$.

But WAIT! From out tables, $L\{e^{at}\} = \frac{1}{s-a}$. Therefore,

$$L\{g(t)\} = \frac{1}{s+1} .$$

Check this out.

$$Q(s) = F(s)G(s)$$

The Laplace transform of a convolution is equal to the product of the Laplace transforms.

$$L\{f(t) * g(t)\} = F(s)G(s)$$

$$L\{f(t) * g(t)\} = L\{f(t)\}L\{g(t)\}$$

$$f(t) * g(t) = \int_0^t f(u)g(t-u)du \qquad\qquad F(s)G(s)$$

R6. Convolution from Power Series.

Take two power series: $A(x) = \sum_{n=o}^{\infty} a_n x^n$ and $B(x) = \sum_{n=o}^{\infty} b_n x^n$.

The capital letters are in a world similar to Laplace-transform space when compared to their respective little letters. The little letters refer to our world.

Multiply the big ones in transform space. Then we must have some sort of convolution for the little "guys." Note that we chose a different summation index for each. If we did not, we would only get the diagonal terms when $k = l$.

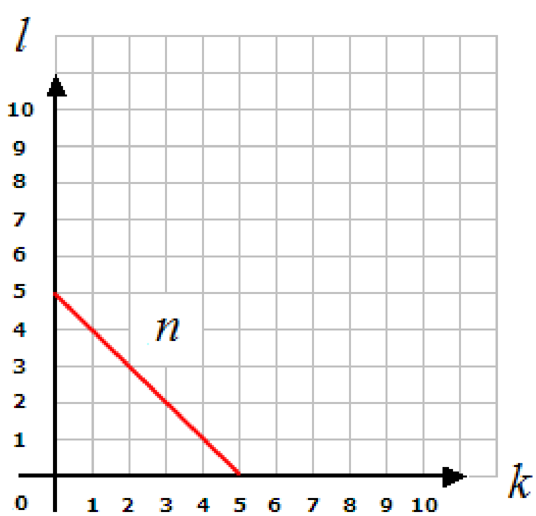


Image Grid from www.helpingwithmath.com

$$A(x)B(x) = \sum_{k=0}^{\infty} a_k x^k \sum_{l=0}^{\infty} b_l x^l$$

$$A(x)B(x) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_k b_l x^{k+l}$$

$$A(x)B(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} x^n$$

Let $n = k + l$. We will sum k from 0 to n and then n from 0 to infinity to do the job.

$$A(x)B(x) = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} a_k b_{n-k} \right] x^n = \sum_{n=0}^{\infty} c_n \, x^n = C(x)$$

The new little "guys" are related to old as follows.

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

This is the convolution - the discrete version. Let's move to the continuous case.

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \, \Delta k \quad \text{since} \quad \Delta k = 1$$

Change delta to d, rip off indexes (promoting to continuous variables), and change the summation sign into a "snake."

$$c(n) = \int_0^n a(k) b(n-k) dk$$

Now replace n with t and k with u.

$$c(t) = \int_0^t a(u) b(t-u) du \, .$$

Oh, what the heck, replace "a" with "f" and "b" with "g" and you have

$$f(t) * g(t) = \int_0^t f(u) \, g(t-u) \, du$$

THE CONVOLUTION!

THE END