Theoretical Physics
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Chapter X Notes. Einstein and the Precession of the Perihelion

X1. Gravitational Effect on Time.

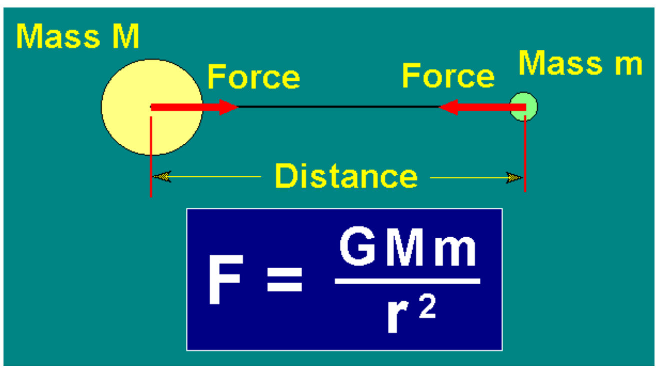
Albert Einstein (1879-1955)
Schweizerische Landesbibliothek

Einstein photo from 1912, three years before arriving at General Relativity.

In our last chapter we derived the following formula relating a bottom clock on the surface of the Earth with a clock at altitude z.

$$T_{bottom} = T_{top}(1 - \frac{gz}{c^2})$$

The clock in the gravitational field runs slower. We would like to arrive at a similar equation for a clock at a distance r from mass M compared to a clock at infinity. We let a photo "fall" from infinity to the distance r. We need the gravitational potential energy for the mass M.



$$F(r) = -\frac{GMm}{r^2} \qquad F(r) = -\frac{dU(r)}{dr} \qquad U(r) = -\frac{GMm}{r}$$

Use Hooke's Law as the analogy. Pull opposite to the spring to stretch it. Then your work is your potential energy. Note force is minus the derivative of the potential energy.

$$F(x) = -kx \qquad W = \int_0^x kx dx = \frac{1}{2}kx^2 = U(x) \qquad F(x) = -\frac{dU}{dx}$$

Photon traveling down to star: $hf_{top} + 0 = hf_{bottom} - \frac{GMm}{r}$

Now use $E = mc^2$ to substitute for mass $m = m_{bottom} = \frac{E_{bottom}}{c^2} = \frac{hf_{bottom}}{c^2}$.

$$hf_{top} + 0 = hf_{bottom} - \frac{GM}{r}\frac{hf_{bottom}}{c^2}$$

$$f_{top} = f_{bottom}(1 - \frac{GM}{c^2 r})$$

Since $T = \frac{1}{f}$, we flip everything to get $T_{top} = \frac{T_{bottom}}{1 - \frac{GM}{c^2 r}}$.

$$T_{bottom} = T_{top}(1 - \frac{GM}{c^2 r})$$

$$\boxed{dt' = (1 - \frac{GM}{c^2 r})\, dt}$$

The unprimed time interval is the one at infinity - free of the gravitational effect.

NOTE: The usual of the classical potential energy makes this calculation a semiclassical one. One needs to use the full formalism of general relativity to find exact relations in general relativity. We will arrive at exact results in the next section by luck.

X2. Gravitational Effect on Spacetime.

Recall the special relativistic effects on space and time. Remember these below?

Lorentz Contraction: $L = L_o\sqrt{1-\dfrac{v^2}{c^2}}$ Time Dilation: $T = \dfrac{T_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$

The ones without the "0" refer to the "lab" frame at rest watching the moving one.

$$L_{lab} = L_o\sqrt{1-\frac{v^2}{c^2}} \qquad T_{lab} = \frac{T_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

Compare these with

$$dt' = (1-\frac{GM}{c^2 r})\,dt$$

The unprimed time is analogous to the lab frame. The lab frame watches the moving frame that contains the rod and clock. Similarly it is the frame at infinity that watches things in the vicinity of the gravitational mass. So let's write the relativity equations with notation where the unprimed refers to the lab.

$$L = L'\sqrt{1-\frac{v^2}{c^2}} \qquad T = \frac{T'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$L' = \frac{L}{\sqrt{1-\frac{v^2}{c^2}}} \qquad T' = T\sqrt{1-\frac{v^2}{c^2}}$$

Now compare special relativity to general relativity.

Special: $$L' = \frac{L}{\sqrt{1-\frac{v^2}{c^2}}} \qquad T' = T\sqrt{1-\frac{v^2}{c^2}}$$

General: ? $$dt' = (1-\frac{GM}{c^2 r})\,dt$$

The analogy suggests that space will be affected by gravity. This inspires us to write the following.

General: $$dr' = \frac{1}{1-\frac{GM}{c^2 r}}dr \qquad dt' = (1-\frac{GM}{c^2 r})\,dt$$

Note that $\frac{GM}{c^2 r} \ll 1$. Then, $(1-\frac{GM}{c^2 r})^2 = 1-\frac{2GM}{c^2 r} + higher\ order$

$$(dr')^2 = \frac{1}{1-\frac{2GM}{c^2 r}}(dr)^2 \qquad (dt')^2 = (1-\frac{2GM}{c^2 r})\,dt^2$$

Do you remember that the invariant in special relativity is the following?

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

Then by analogy, for general relativity we have for spherical coordinates this.

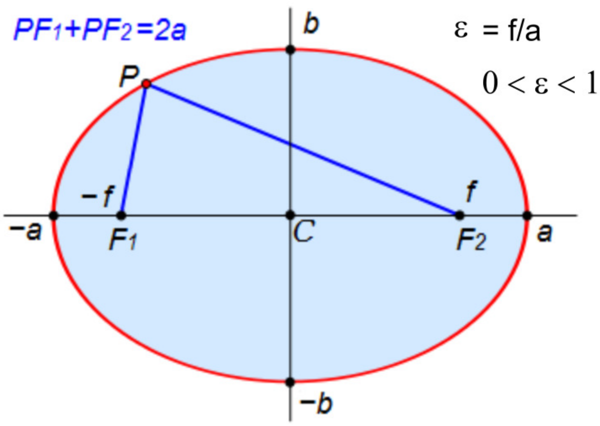
$$ds^2 = (1-\frac{2GM}{c^2 r})c^2 dt^2 - \frac{1}{(1-\frac{2GM}{c^2 r})}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

This is called the line element. The above turns out to be the exact form in general relativity for a spherical mass. Note that far away, you get the special relativistic case.

X3. Kepler's Three Laws.

Kepler's First Law: The Law of Ellipses.

Planets travel around the Sun along ellipses where the Sun is at a focus.



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

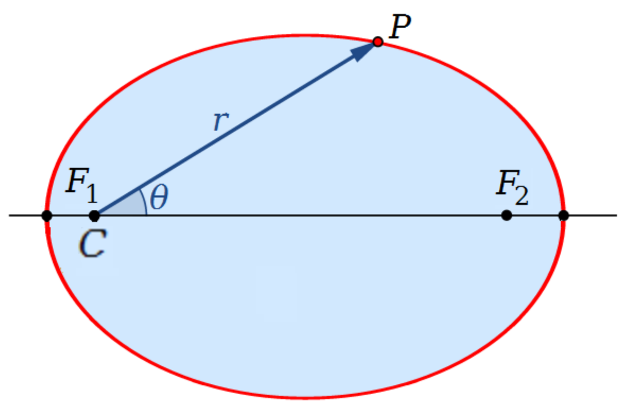
String length: $PF_1 + PF_2 = 2a$

Eccentricity ε : $a\varepsilon \equiv f$

P at (0,b): $PF_1 = PF_2 = a$

Then $a^2 = f^2 + b^2$ $a^2 = a^2\varepsilon^2 + b^2$ $b^2 = (1-\varepsilon^2)a^2$

Shift F_1 to C: $\dfrac{(x-\varepsilon a)^2}{a^2}+\dfrac{y^2}{b^2}=1$



Polar Coordinates: $r(\theta)$

$x = r\cos\theta$ and $y = r\sin\theta$

Hold on the substitution for now.

First, it's time for Algebra City.

From $\dfrac{(x-\varepsilon a)^2}{a^2}+\dfrac{y^2}{b^2}=1$ we can show $r=\dfrac{a(1-\varepsilon^2)}{1-\varepsilon\cos\theta}$.

$$\frac{(x-\varepsilon a)^2}{a^2}+\frac{y^2}{(1-\varepsilon^2)a^2}=1$$

$$(x-\varepsilon a)^2 + \frac{y^2}{1-\varepsilon^2} = a^2$$

$$x^2 - 2a\varepsilon x + \varepsilon^2 a^2 + \frac{y^2}{1-\varepsilon^2} = a^2$$

$$x^2 - 2a\varepsilon x + \frac{y^2}{1-\varepsilon^2} = a^2 - \varepsilon^2 a^2$$

$$x^2 - 2a\varepsilon x + \frac{y^2}{1-\varepsilon^2} = a^2(1-\varepsilon^2)$$

$$x^2 + \frac{y^2}{1-\varepsilon^2} = a^2(1-\varepsilon^2) + 2a\varepsilon x$$

$$(1-\varepsilon^2)x^2 + y^2 = a^2(1-\varepsilon^2)^2 + 2a\varepsilon x(1-\varepsilon^2)$$

$$x^2 - \varepsilon^2 x^2 + y^2 = a^2(1-\varepsilon^2)^2 + 2a\varepsilon x(1-\varepsilon^2)$$

$$x^2 + y^2 = a^2(1-\varepsilon^2)^2 + 2a\varepsilon x(1-\varepsilon^2) + \varepsilon^2 x^2$$

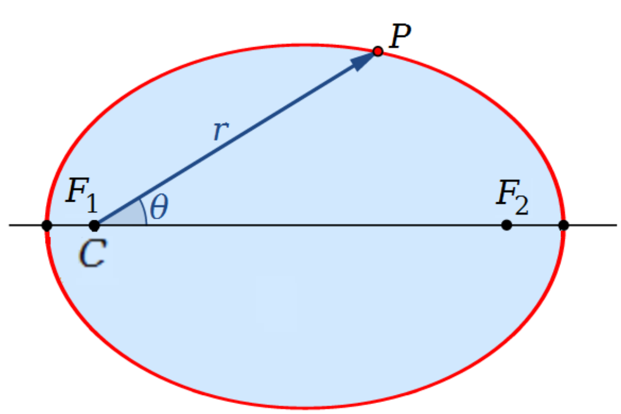
$$r^2 = \left[a(1-\varepsilon^2) + \varepsilon x\right]^2 \qquad r = a(1-\varepsilon^2) + \varepsilon x > 0$$

$$r - \varepsilon x = a(1-\varepsilon^2)$$

$$r - \varepsilon r\cos\theta = a(1-\varepsilon^2)$$

$$r(1-\varepsilon\cos\theta) = a(1-\varepsilon^2)$$

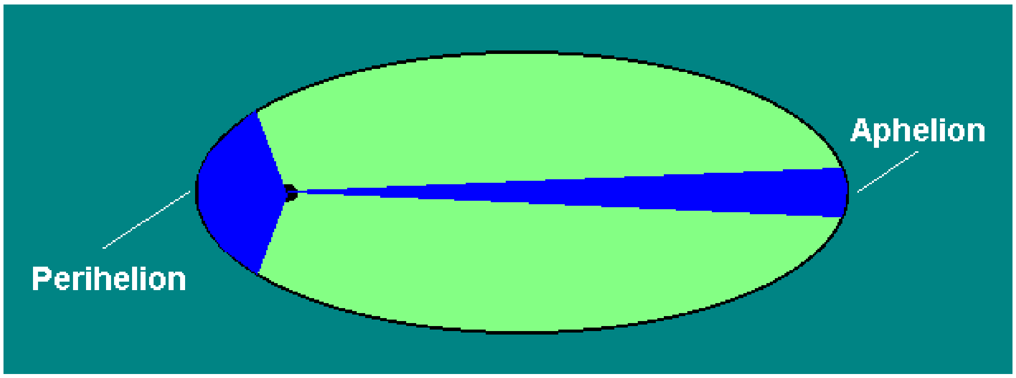
$$r = \frac{a(1-\varepsilon^2)}{1-\varepsilon\cos\theta}$$



Kepler's Second Law: The Law of Areas.

The planet sweeps out equal areas in equal times.

The perihelion is the closest distance and the aphelion is the farthest.



Kepler's Second Law states that the areal velocity is constant. The differential area is arrived at using the formula for a triangle on the thin pie slice.

$$dA = \frac{1}{2} R(R\,d\theta) = \frac{1}{2} R^2 d\theta$$

$$\frac{dA}{dt} = \frac{1}{2} R^2 \frac{d\theta}{dt} = const$$

Kepler's Third Law: The Law of Periods.

The cube of the semi-major axis is proportional to the square of the period.

For a Circular Orbit: $a = R$ $\qquad \frac{GMm}{R^2} = m\frac{v^2}{R} \qquad \frac{GMm}{R^2} = m\frac{1}{R}\left[\frac{2\pi R}{T}\right]^2$

$$\frac{GM}{R} = \frac{4\pi^2 R^2}{T^2} \qquad GMT^2 = 4\pi^2 R^3$$

$$R^3 = \frac{GM}{4\pi^2} T^2$$

X4. Precession of the Perihelion. Reference: S. Cornbleet, "Elementary Derivation of the Advance of the Perihelion of a Planetary Orbit," *American Journal of Physics* **61**, pp. 650-651 (1993). Note that his theta angle is a supplement to ours.

Kepler's Second Law states that the areal velocity is constant. The differential area is arrived at using the formula for a triangle on the thin pie slice.

$$dA = \frac{1}{2}R(Rd\theta) = \frac{1}{2}R^2 d\theta$$

Note that R is the distance to the planet. Then Kepler's Second Law is

$$\frac{dA}{dt} = \frac{1}{2}R^2\frac{d\theta}{dt} = const\,.$$

We have to be careful in general relativity since the radial dimension is distorted. So we back up and write the differential area as an integral with respect to the radial variable.

$$dA = \int_0^R dr(rd\theta) = \int_0^R rdrd\theta = \left.\frac{r^2}{2}\right|_0^R d\theta = \frac{R^2}{2}d\theta$$

In general relativity our guide derives from the line element.

$$ds^2 = (1-\frac{2GM}{c^2r})c^2dt^2 - \frac{1}{(1-\frac{2GM}{c^2r})}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2$$

$$dA' = \int_0^R dr'(rd\theta) = \int_0^R rdr'd\theta$$

$$dr' = (1-\frac{2GM}{c^2r})^{-\frac{1}{2}}dr \approx (1+\frac{GM}{c^2r})dr$$

The latter result is the one we arrived at directly from a semiclassical discussion in the previous section. The we find

$$dA' = \int_0^R (r+\frac{GM}{c^2})\,drd\theta$$

$$dA' = \int_0^R (r + \frac{GM}{c^2})\, drd\theta$$

$$dA' = (\frac{r^2}{2} + \frac{GMr}{c^2})\Big|_0^R\, d\theta$$

$$dA' = (\frac{R^2}{2} + \frac{GMR}{c^2})d\theta$$

$$dA' = \frac{R^2}{2}(1 + \frac{2GM}{c^2 R})d\theta$$

Now for the areal velocity in the distorted spacetime.

$$\frac{dA'}{dt'} = \frac{R^2}{2}(1 + \frac{2GM}{c^2 R})\frac{d\theta}{dt'}$$

$$ds^2 = (1 - \frac{2GM}{c^2 r})c^2 dt^2 - \frac{1}{(1 - \frac{2GM}{c^2 r})}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

$$dt' = (1 - \frac{2GM}{c^2 r})^{\frac{1}{2}} dt \approx (1 - \frac{GM}{c^2 r})dt$$

But in $\frac{dA'}{dt'} = \frac{R^2}{2}(1 + \frac{GM}{c^2 R})\frac{d\theta}{dt'}$ we are at $r = R$ and need

$$\frac{d}{dt'} = \frac{1}{(1 - \frac{GM}{c^2 R})}\frac{d}{dt} \approx (1 + \frac{GM}{c^2 R})\frac{d}{dt}.$$

Then,

$$\frac{dA'}{dt'} = \frac{R^2}{2}(1+\frac{2GM}{c^2R})\frac{d\theta}{dt'} = \frac{R^2}{2}(1+\frac{2GM}{c^2R})(1+\frac{GM}{c^2R})\frac{d\theta}{dt}$$

$$\frac{dA'}{dt'} = \frac{R^2}{2}(1+\frac{3GM}{c^2R})\frac{d\theta}{dt} + higher\ order$$

The $\frac{R^2}{2}$ term is the classical term. Now we have an extra term with it.

Classical $\quad \frac{dA}{dt} = \frac{1}{2}R^2\frac{d\theta}{dt}$

General Relativity $\quad \frac{dA'}{dt'} = \frac{R^2}{2}(1+\frac{3GM}{c^2R})\frac{d\theta}{dt}$

Comparing the two, we have an effective angle $d\theta' = (1+\frac{3GM}{c^2R})d\theta$.

$$R = \frac{a(1-\varepsilon^2)}{1-\varepsilon\cos\theta} \qquad \frac{1}{R} = \frac{1-\varepsilon\cos\theta}{a(1-\varepsilon^2)}$$

$$\int_0^{2\pi}(1+\frac{3GM}{c^2R})d\theta = \int_0^{2\pi}(1+\frac{3GM}{c^2}\frac{1-\varepsilon\cos\theta}{a(1-\varepsilon^2)})d\theta$$

$$\Delta\theta' = \int_0^{2\pi}d\theta + \frac{3GM}{c^2a(1-\varepsilon^2)}\int_0^{2\pi}d\theta - \frac{3GM}{c^2a(1-\varepsilon^2)}\varepsilon\int_0^{2\pi}\cos\theta d\theta$$

Note that the last integral is zero: $\int_0^{2\pi}\cos\theta d\theta = \sin\theta\Big|_0^{2\pi} = 0-0=0$

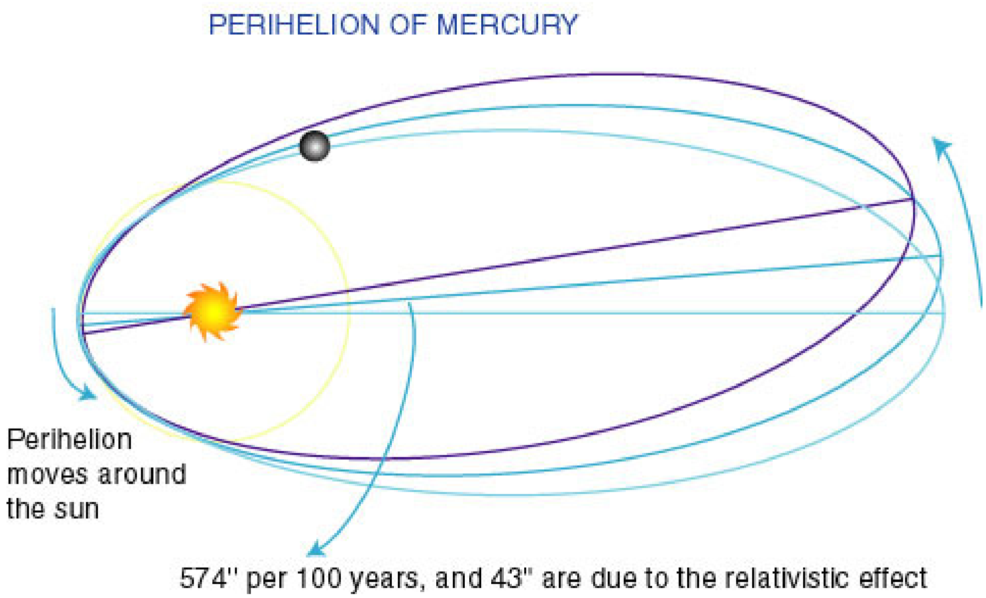
The first integral is the standard closed orbit, but the second term gives the advance.

$$\delta = \frac{6\pi GM}{c^2a(1-\varepsilon^2)}$$

X5. The 43" per Century. In our previous section we found that for each orbit, the planet advances by the angle

$$\delta = \frac{6\pi GM}{c^2 a(1-\varepsilon^2)}.$$

This is referred to as the advance of the perihelion. However, you can think of the entire orbit as rotating by this amount each revolution of the planet.



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Excluding "spinning-top physics effects," astronomers observed that Mercury advances 574" per century. All but 43" could be accounted for by Newtonian effects of other planets. The 43"/century remained a mystery for decades.

$$\delta = \frac{6\pi GM}{c^2 a(1-\varepsilon^2)}$$

$$G = 6.6742x10^{-11} N(m/kg)^2 \qquad c = 299,792,458\, m/s \text{ (EXACT)}$$

$$M_{sun} = 1.98892x10^{30} kg \qquad a_{mercury} = 57.91x10^9 m \text{ (semimajor axis)}$$

$$\varepsilon_{mercury} = 0.20563$$

$$\delta = \frac{6\pi(6.6742x10^{-11}\,kg\cdot m/s^2)(m/kg)^2)(1.98892x10^{30}\,kg)}{(299,792,458\,m/s)^2(57.9091x10^9\,m)(1-0.20563^2)}$$

$$\delta = 5.01987x10^{-7}\,\frac{kg\cdot\frac{m}{s^2}\cdot\frac{m^2}{kg^2}\cdot kg}{\frac{m^2}{s^2}\cdot m} = 5.01987x10^{-7}\,radians$$

Revolutions per century:

$$N = \frac{100\;years}{87.9391\,days}\cdot\frac{365.242199\,days}{1\;year} = 415.3354$$

$$\Delta = N\delta = 2.08493x10^{-4}\;rad$$

$$\Delta = 2.08493x10^{-4}\;rad\cdot\frac{180°}{\pi\,rad}\cdot\frac{3600"}{1°}$$

$$\Delta_{GR} = 43.00"/\,century$$