e e e

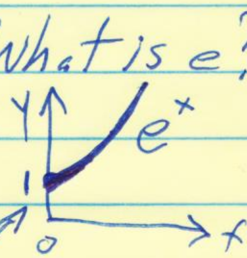
B1

B. What is e? Euler, Integral Tricks — include e during integration

Jan 16, 2020

Class focuses on e

B1. What is e?



Slope is 1 at x=0

$$\frac{de^x}{dx} \equiv e^x$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$(a+b)^n = a^n b^0 + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1.2}a^{n-2}b$$

$$+ \frac{n(n-1)(n-2)}{1.2.3}a^{n-3}b^3 + \dots$$

apples bananas

$$\left(1+\frac{x}{n}\right)^n = 1 + \frac{n}{1}\left(\frac{x}{n}\right) + \frac{n(n-1)}{1.2}\left(\frac{x}{n}\right)^2 + \frac{n(n-1)(n-2)}{1.2.3}\left(\frac{x}{n}\right)^3 + \dots$$

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = e^x$$

B2. Economics and e   Let x=1

Let n=1 $(1+1)^1 = 1+1 = 2$ \$2.00   100% interest

Let n=2 $(1+\frac{1}{2})(1+\frac{1}{2}) = \frac{3}{2}\cdot\frac{3}{2} = \frac{9}{4} = 2.25$ compound 6 mo.

Let n=3 $(1+\frac{1}{3})^3 = \left(\frac{4}{3}\right)^3 = \frac{64}{27} = 2.37$ 4 mo.

n=4 $(1+\frac{1}{4})^4 = \left(\frac{5}{4}\right)^4 = \frac{625}{256} = 2.44$ 3 mo.

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

Let x=1 $\Rightarrow$ $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.71828\dots$

Compound moment by moment

Irrational, Transcendental $\boxed{\$2.72}$

No ratio of integers

Not Algebraic

Solution of algebraic equation — Algebraic

Polynomial with integer coefficients

$x^2 - 2 = 0$
$x = \sqrt{2}$
$\Rightarrow x \to \sqrt{2}$

But not transcendental

Best Known e, π

also irrational

MORE on Algebraic Numbers

Algebraic no. – solution of polynomial eq. with integer coefficients

2 is algebraic – solution to $x-2=0$

1/2 " " – " " $2x-1=0$

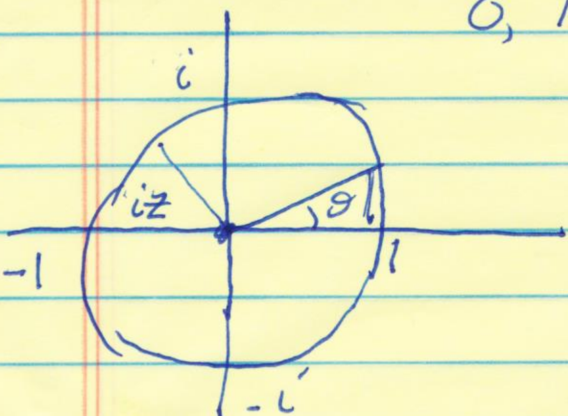
$\sqrt{2}$ " " " " $x^2-2=0$

B2

Transcendental no. is not algebraic

B3. Our Jewel: Euler's Formula

$0, 1, i, \pi, e$



$z = \cos\theta + i\sin\theta$

Rotated by 90°

$$\frac{dz}{d\theta} = -\sin\theta + i\cos\theta = iz$$

$$\Rightarrow z = e^{i\theta} = \cos\theta + i\sin\theta$$

B4. Our Opal: Euler's Identity $e^{i\pi} = -1$

$$\Longrightarrow e^{i\pi} + 1 = 0$$

Another Derivation $z^2 = e^{2i\theta} \Rightarrow$ Double angle

see text for 45° case

$$\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2 = i$$

why?

Earlier

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

$$e^{i\varepsilon} = 1 + i\varepsilon = 1 + i\frac{\pi}{n}$$

$$-1 = \lim_{n\to\infty}\left(1 + \frac{i\pi}{n}\right)^n$$

$$-1 = e^{i\pi} \Rightarrow e^{i\pi} + 1 = 0$$

B5. Integral Trick 1: Real-Imaginary Trick

$$\int_0^\infty e^{-ax}\cos kx\, dx$$

$$\int_0^\infty e^{-ax}\underbrace{(\cos kx + i\sin kx)}_{e^{ikx}}\, dx$$

$$= \int_0^\infty e^{-ax+ikx}\, dx = \left.\frac{e^{-ax+ikx}}{-a+ik}\right|_0^\infty = \frac{0-1}{-a+ik}$$

$$= \frac{1}{a-ik} = \frac{a+ik}{a^2+k^2}$$

Real $\Rightarrow \dfrac{a}{a^2+k^2}$ Im $\Rightarrow \dfrac{k}{a^2+k^2}$

B6. Integral Trick 2: Polar

$$I=\int_{-\infty}^{\infty} e^{-x^2}dx \qquad \sqrt{\pi}$$

$$I^2=\int_{-\infty}^{\infty} e^{-x^2}dx\int_{-\infty}^{\infty} e^{-y^2}dy=\int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} e^{-r^2}\,r\,dr\,d\theta$$

$$I^2= 2\pi\left.\frac{e^{-r^2}}{-2}\right|_0^{\infty}=2\pi\left[0-\left(\frac{1}{-2}\right)\right]=\pi$$

$$I=\int_{-\infty}^{\infty} e^{-ax^2}dx=\sqrt{\frac{\pi}{a}} \qquad a>0$$

B7. Integral Trick 3: Derivative Trick

$$I=\int_{-\infty}^{\infty} e^{-ax^2}dx=\sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^{\infty} x^2e^{-ax^2}dx=-\frac{d}{da}\sqrt{\frac{\pi}{a}}=\frac{1}{2a}\sqrt{\frac{\pi}{a}}$$

Can Continue

B8. Integral Trick 4: Completing The Square

$$I=\int_{-\infty}^{\infty} e^{-ax^2+ikx}dx \qquad a>0$$

Let $b=ik$

$$ax^2-bx=a\left(x-\frac{b}{2a}\right)^2-\frac{b^2}{4a}$$

$$I=\int_{-\infty}^{\infty} e^{-a(x-\frac{b}{2a})^2}e^{b^2/4a}dx=e^{b^2/4a}\underbrace{\int_{-\infty}^{\infty} e^{-az^2}dz}_{\sqrt{\frac{\pi}{a}}}$$

$$I=e^{b^2/4a}\sqrt{\frac{\pi}{a}}$$

finally $b=ik \Rightarrow I=e^{-k^2/4a}\sqrt{\frac{\pi}{a}}$