February 13, 2020

Class J. Spinors

Example

Complex number

$z = 3+4i$

$z^* = 3-4i$

complex conjugate

J1. Matrix Fun $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

1. Trace $\mathrm{Tr}\,A = a+d$
2. Transpose $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ $\mathrm{Tr}\,A^T = \mathrm{Tr}\,A$
3. Complex Conjugate $A^* = \begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix}$ In pure math $\bar{A}$ is used.
4. Hermitian Conjugate

$$A^\dagger = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} = (A^T)^* = (A^*)^T$$

In pure math a star or H is used.

5. Determinant $\det A \equiv |A| = ad - bc$
6. Inverse Some matrices do not have inverses.

$$A^{-1} \text{ such that } AA^{-1} = I \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Psych out the inverse for 2×2 $\Big\}$ $A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$AA^{-1} = \frac{1}{|A|}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & \overbrace{-ab+ba}^{0} \\ \underbrace{cd-dc}_{0} & -bc+da \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

J2. Unitary Groups $U^\dagger = U^{-1}$

Binary operation Matrix multiplication

1. $U(1)$ 1×1 Matrices

$$U = [u] \qquad U^\dagger = [u^*]$$

$$UU^* = [u][u^*] = [uu^*] = [1]$$

Must be to qualify

$$u = e^{i\theta} \quad \text{works}$$

Closure: $u(\alpha)u(\beta) = e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)} \in U(1)$

Association: ok Good old fashion multiplication.

Identity: $I = u(0) = 1$

Inverse: $u^{-1}(\theta) = e^{-i\theta}$ consistent with $U^\dagger = U^{-1}$

2. $SU(1)$ — Special

$\det U = \det[u] = u = 1$

Subgroup $SU(1) \subset U(1)$

Single matrix $A = [1]$

3. $SU(2)$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ in general from before

Special $\Rightarrow \det A = 1$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Unitary $\Rightarrow A^{\dagger} = A^{-1}$ $\quad A^{\dagger} = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix}$ must be the same

Compare elements of $A^{\dagger}$ and A^{-1} to get $a^* = d \quad c = -b^* \quad |A| = ad - bc = 1$

Real Numbers

$A = \begin{bmatrix} a_r + i a_i & b_r + i b_i \\ -b_r + i b_i & a_r - i a_i \end{bmatrix}$ Write with real + imaginary parts

Imaginary

Note $\underbrace{(a+bi)(a-bi)}_{a^2+b^2}$

$\det A = 1 \Rightarrow a_r^2 + a_i^2 + b_r^2 + b_i^2 = 1$

$A = a_r \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i a_i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b_r \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + i b_i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Would like the last three have i out in front (all similar)

$A = a_r \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} + i a_i \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\sigma_z} + i b_r \underbrace{\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}}_{\sigma_y} + i b_i \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\sigma_x}$

Pauli Matrices

$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ By Defition

σ_z for up + down
σ_x next simplest
σ_y last choice left

Anticommutator $\{A, B\} = AB + BA$

$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \{\sigma_j, \sigma_k\} = 2\delta_{jk} I$ — Kronecker delta

$\varepsilon_{ijk} = \begin{cases} +1 & 123 \text{ cyclic} \to 231, 312 \\ -1 & 132, 321, 213 \\ 0 & \text{any 2 same} \end{cases} \quad [\sigma_j, \sigma_k] = 2i\,\varepsilon_{jkl}\,\sigma_l$ — Levi-Civita Symbol

$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

J3. Eigenvalues $\uparrow = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\downarrow = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\sigma_z \uparrow = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = +\uparrow$ — eigenvalue is +1

eigenvector

Got same vector back

$\sigma_z \downarrow = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\downarrow$

Minus sign eigenvalue

Consider $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Find eigenvectors

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \lambda \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$ have to get same vector back

$$\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$ often writen with just the 0

set determinant to zero

$$\begin{bmatrix} -\lambda c_1 + c_2 \\ c_1 - \lambda c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -\lambda c_1 + c_2 &= 0 \\ c_1 - \lambda c_2 &= 0 \end{aligned}\right\} \quad -\lambda(\lambda c_2) + c_2 = 0$$

$$\underbrace{(-\lambda^2 + 1)}\, c_2 = 0$$

Short cut for future use $\left\{ \det\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \right.$

$\lambda^2 = 1$ $\lambda = \pm 1$ eigenvalues

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = (+1)\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Rightarrow \begin{bmatrix} c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$c_1 = c_2$ Normalize: $u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Find the eigenvector that goes with the minus 1 eigenvalue

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = (-1)\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$\begin{bmatrix} c_2 \\ c_1 \end{bmatrix} = -\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ $c_2 = -c_1$ $V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$