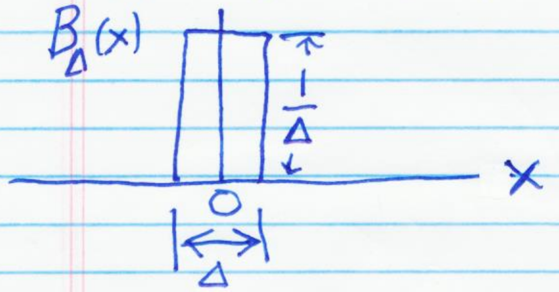
Class N. The Dirac Delta Function

N1. The Dirac Delta Function

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases} \quad \text{with} \int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

The Box Function

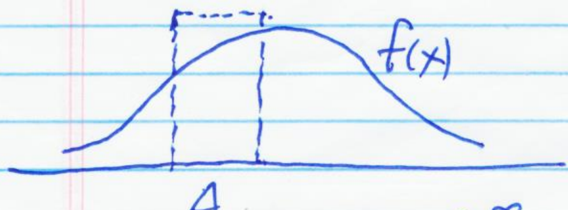


$B_\Delta(x) = 0$ outside the Δ region

$B_\Delta(x) = \frac{1}{\Delta}$ inside the Δ region

$$\int_{-\infty}^{\infty} B_\Delta(x)\,dx = \frac{1}{\Delta}\Delta = 1$$

$$\lim_{\Delta \to 0} B_\Delta(x) = \delta(x)$$



$$\int_{-\infty}^{\infty} B_\Delta(x) f(x)\,dx \overset{MVT}{=} \frac{1}{\Delta}\underbrace{f(c)\Delta}_{\text{Area}} = f(c)$$

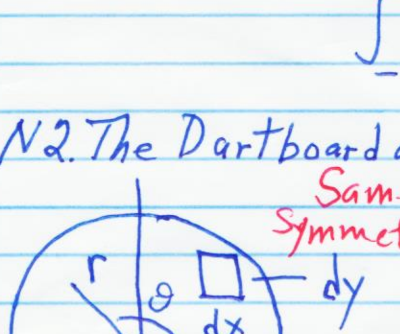
$$\lim_{\Delta \to 0} \int_{-\infty}^{\infty} B_\Delta(x) f(x)\,dx \overset{MVT}{=} \lim_{\Delta \to 0} f(c) = f(0)$$

Sifting Result $\int_{-\infty}^{\infty} \delta(x) f(x)\,dx = f(0)$

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\,dx = f(a)$$

the spike is now at $x=a$
Shifted to the right

N2. The Dartboard and the Gaussian



(r,θ)
Polar coordinates

$x = r\cos\theta \quad y = r\sin\theta$

Same P ↘ statistics
Symmetry $P(x)\,dx\,P(y)\,dy$

Prob. between x and x+dx

Probability between y and y+dy

$\sim g(r)\,dr$ ribbon through dxdy region

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$g(r) = P(x)P(y)$ independent of angle θ

$$\frac{\partial g(r)}{\partial \theta} = 0 \qquad \frac{\partial P(x)}{\partial \theta} P(y) + P(x)\frac{\partial P(y)}{\partial \theta} = 0$$

$x = r\cos\theta \qquad y = r\sin\theta$

$\frac{\partial x}{\partial \theta} = -r\sin\theta = -y \qquad \frac{\partial y}{\partial \theta} = r\cos\theta = x$

$$\frac{dP(x)}{dx}\frac{\partial x}{\partial \theta}P(y) + P(x)\frac{dP(y)}{dy}\frac{\partial y}{\partial \theta} = 0$$

$\llcorner -r\sin\theta = -y \qquad \llcorner r\cos\theta = x$

$$\frac{dP(x)}{dx}\,y\,P(y) = \frac{dP(y)}{dy}\,x\,P(x)$$

$$\frac{1}{xP(x)}\frac{dP(x)}{dx} = \frac{1}{yP(y)}\frac{dP(y)}{dy} = C$$

As $x + y$ are independent

Remember

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}dx = \sqrt{\frac{\pi}{\alpha}}$$

$$\int \frac{1}{P(x)}dP(x) = \int C\,x\,dx$$

another constant

Let $\alpha = k/2$ for $P(x)$

$$\ln P(x) = \frac{Cx^2}{2} + c$$

A

Gaussian

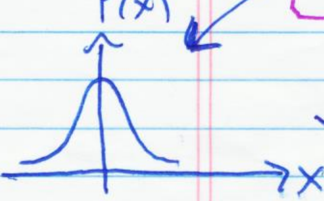
$$P(x) = e^{Cx^2/2}e^{c}$$

$\boxed{P(x) = Ae^{-\frac{k}{2}x^2}}$

$\hookrightarrow C = -k \quad k>0$ to avoid blowing up as $x \to \infty$

Should get zero for missing the bull's eye that badly.

$P(x)$



$$\int_{-\infty}^{\infty} P(x)\,dx = A\int_{-\infty}^{\infty} e^{-\frac{k}{2}x^2}dx = 1$$

$$\sqrt{\frac{\pi}{k/2}} = \sqrt{\frac{2\pi}{k}} \Rightarrow A = \sqrt{\frac{k}{2\pi}}$$

σ^2 or σ

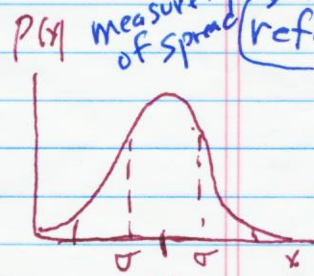
Variance

Dispersion refers to Spread

$P(x)$ measures of spread

$$\sigma^2 = \int_{-\infty}^{\infty} x^2P(x)\,dx$$

Spread $\quad \frac{d}{d\alpha}\alpha^{-1/2} = -\frac{1}{2}\alpha^{-3/2}$



$$\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2}dx = -\frac{d}{d\alpha}\int_{-\infty}^{\infty} e^{-\alpha x^2}dx = -\frac{d}{d\alpha}\sqrt{\frac{\pi}{\alpha}} = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}$$

$$\alpha = \frac{k}{2} \Longrightarrow \frac{1}{k}\sqrt{\frac{2\pi}{k}}$$

$$\sigma^2 = A\int_{-\infty}^{\infty} x^2 e^{-\frac{k}{2}x^2}dx = \sqrt{\frac{k}{2\pi}}\,\frac{1}{k}\sqrt{\frac{2\pi}{k}} = \frac{1}{k}$$

$A\downarrow$

68-95-99.7 Rule

$\pm\sigma$

$\pm 2\sigma$ 95%

$\pm 3\sigma$ 99.7%

$$k = \frac{1}{\sigma^2} \qquad P(x) = \sqrt{\frac{1}{2\pi\sigma^2}}\,e^{-\frac{x^2}{2\sigma^2}}$$

μ average or mean

For average $\mu \neq 0$ Shift to right.

N3. Statistics $P(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\,e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

See Gaussian in Text.

σ is the standard deviation $\quad \sigma^2$ Variance

N4. Delta Sequence of Gaussians

$$\delta_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

See delta sequence graphs in text.

Mathematicians like continuous functions.

└ do not feel comfortable with our original $\delta(x)$

$$\delta(x) = \lim_{\sigma \to 0} \delta_\sigma(x)$$

└ standard deviation goes to zero
no spread

And Gaussians shoots upward due to $-\frac{x^2}{2\sigma^2}$ in exponential.

Remember this kind of result?

$$\hookrightarrow \int_{-\infty}^{\infty} e^{-\alpha k^2 + ikx}\, dk = \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{x^2}{4\alpha}}$$

$$\alpha \to \frac{\sigma^2}{2}$$

$$\int_{-\infty}^{\infty} e^{-\frac{\sigma^2}{2}k^2} e^{ikx}\, dk = \sqrt{\frac{2\pi}{\sigma^2}}\, e^{-\frac{x^2}{2\sigma^2}}$$

$\sigma \to$ small gives tall + thin

Thin

Need this factor

Tall

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{\sigma^2}{2}k^2} e^{ikx}\, dk = \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{x^2}{2\sigma^2}}}_{\delta_\sigma(x)}$$

Now Let $\sigma \to 0$

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\, dk$$

What?

cosines + sines

You can't integrate cosines + sines to get a delta function.

Now here's another reason physicists make mathematicians nervous.

Interpret as limiting sequence of Gaussians.

If you are nervous, stick a Gaussian back in there.