Class S Cauchy Integral Formula

S1. Cauchy-Riemann Conditions

Complex constant $a+ib$ $i=\sqrt{-1}$ a, b Real

" variable $z = x+iy$ $\{x, y \in R\}$

function $f(z) = u(x,y) + i\,v(x,y)$ $\{u, v \in R\}$

Q1. Is differentiation well-defined for complex functions?

$$\frac{\Delta f}{\Delta z} = \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y} \nearrow \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \text{ along } x$$

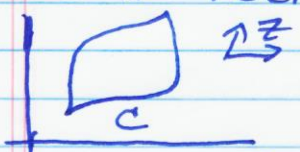
$$\searrow \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \text{ along } y$$

For the derivative to be unique

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Q2. What about integration paths?

S2. Green's Theorem



$$\oint f(z)\,dz = 0$$ Closed path

$$\oint (u+iv)(dx+i\,dy) = 0$$

We demand these to be true.

Form

$$I = \oint B_x\,dx + B_y\,dy \Leftarrow \oint (u\,dx - v\,dy) + i\oint (v\,dx + u\,dy) = 0$$

$\hookrightarrow$ Stoke's Theorem $\oint \vec{B}\cdot d\vec{l} = \iint_A (\nabla\times\vec{B})\cdot d\vec{A}$ ($d\vec{A}$ = $\hat{k}\,dx\,dy$)

We are in a plane

Green's Theorem Special case

$$\iint_A \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) dx\,dy$$

$$-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Amazing! Same results.

$f(z)$ is Analytic when Cauchy-Riemann Conditions are met

Green's Theorem usual form

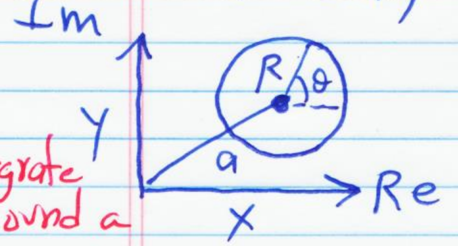
$$\oint_C (L\,dx + M\,dy) = \iint_A \left[\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right] dx\,dy$$

a special case of Stoke's Theorem

Stoke's Theorem in a plane.

Now for complex plane magic!

S3. Cauchy Integral Formula



Integrate around a scary singularity.

$$I = \oint \frac{1}{z-a}\,dz$$ counterclockwise; $\frac{1}{z-a}$: singularity

Points on Circle: $z = a + \underbrace{R\cos\theta + iR\sin\theta}_{Re^{i\theta}}$

$$z - a = Re^{i\theta} \qquad dz = iRe^{i\theta}\,d\theta$$

$$I = \oint \frac{1}{z-a}\,dz = \int_0^{2\pi} \frac{iRe^{i\theta}}{Re^{i\theta}}\,d\theta = \int_0^{2\pi} i\,d\theta = 2\pi i$$

Consider $I = \oint \frac{f(z)}{z-a}\,dz$ where $f(z)$ has no singularities.

Add + subtract the same thing.

$$I = \oint \frac{f(z)-f(a)}{z-a}\,dz + \underbrace{f(a)\oint \frac{1}{z-a}\,dz}_{\to\ 2\pi i f(a)}$$

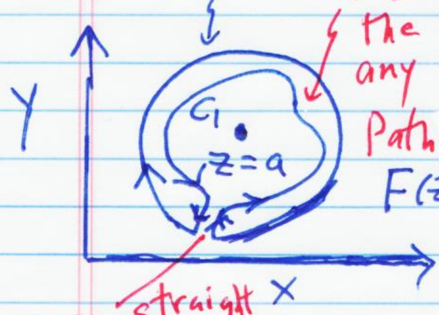
$$\left|\oint \frac{f(z)-f(a)}{z-a}\,dz\right| = \oint \left|\frac{f(z)-f(a)}{R}\right| ds \le \frac{M}{R}(2\pi R) = 2\pi M$$

M maximum $|f(z)-f(a)|$ along circumference

$R \to 0 \Rightarrow |f(z)-f(a)| \to |f(a)-f(a)| = 0$

$$\boxed{\oint \frac{f(z)}{z-a}\,dz = 2\pi i f(a)}$$

True for any closed path.



usual Circle

C_1 is the any Path.

These straight lines cancel when the narrow gap is closed.

$$F(z) = \frac{f(z)}{z-a} \qquad \underbrace{\oint_C F(z)\,dz}_{\text{No Singularity inside}} = \underbrace{\oint_{C_1} F(z)\,dz}_{\text{Any Path}} + \underbrace{\oint_{C_2} F(z)\,dz}_{-2\pi i f(a)} = 0$$

$C = C_1 + C_2$

$$\oint_{\text{Any}} \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$