April 2, 2020

Class T. Poles + The Residue Theorem

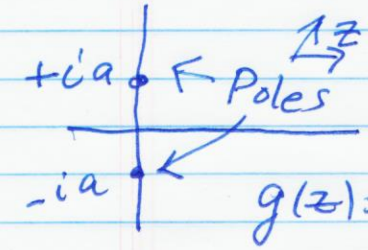
From Last Class Cauchy Integral Formula

T1. Poles

$$\oint_a \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$ ← analytic everywhere

$F(z) = \dfrac{f(z)}{z-a}$ analytic with singularity at $z=a$

Consider $g(z) = \dfrac{1}{z^2+a^2}$ $a>0$



↳ analytic with 2 singularities

$0 = z^2 + a^2 = (z+ia)(z-ia) = 0$

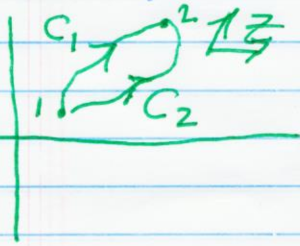
or $z^2 = -a^2$ $z = \pm i a$

$$g(z) = \frac{1}{(z+ia)(z-ia)}$$

T2. Residue

for $f(z)$ analytic everywhere $\oint f(z)\,dz = 0$

Recall



$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$ Path Independence Cauchy-Riemann Conditions

$$\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$

minus ⇒ Coming back $-C_2$ ⇒ Loop

$$\oint F(z)\,dz = 0 \quad \text{closed path}$$

Last class $\displaystyle\oint_a F(z)\,dz = \oint_a \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$

We Want a formal way to get answer from $F(z)$

3 Steps
- i) Clear the singularity $(z-a)F(z) = f(z)$
- ii) Set $z=a$ ⇒ $f(a)$
- iii) Multiply by $2\pi i$ ⇒ $2\pi i f(a)$

$$\oint_a F(z)\,dz = 2\pi i \left[(z-a)F(z)\right]_{z=a}$$

Residue of $F(z)$ at a is

$$\mathrm{Res}(F,a) = \lim_{z\to a}\left[(z-a)F(z)\right]$$

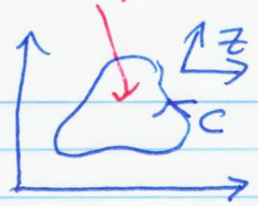
$$\oint_a F(z)\,dz = 2\pi i\, \mathrm{Res}(F,a)$$

No poles inside

What about multiple poles?

T3. The Residue Theorem

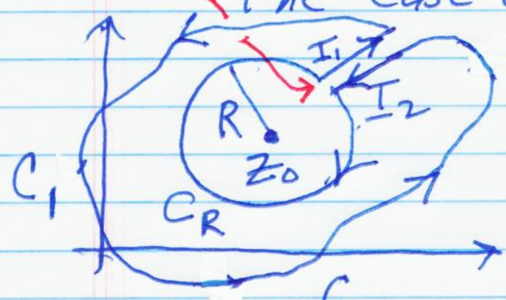
The Case of No Poles



$$\oint F(z)\,dz = 0$$

Start

The Case of One Pole



$$F(z) = \frac{f(z)}{z - z_0}$$

$$\underset{\text{Total}}{C_T} = I_1 + C_1 + I_2 + C_R$$

$$\oint_{C_T} F(z)\,dz = 0$$

no poles inside the area bounded by the path

$$\int_{I_1} F(z)\,dz + \oint_{C_1} F(z)\,dz + \int_{I_2} F(z)\,dz + \oint_{C_R} F(z)\,dz = 0$$

I_1 I_2

I_1, I_2 Cancels as we will close the gap

As Gap closes, we no longer have the cirular C_R segment of our journey as we are back to our start.

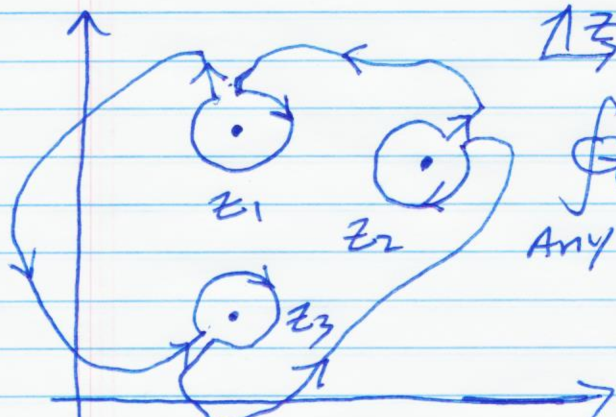
Two Things when gap closes 1) $\oint_{C_R} F(z)\,dz = -2\pi i f(z_0)$ Complete circle

2) C_1 now encloses the pole

$$\oint_{\text{Any}} F(z)\,dz - 2\pi i f(z_0) = 0$$

$$\oint F(z)\,dz = 2\pi i \, Res(F, z_0)$$

The Case of Multiple Poles



$$F(z) = \frac{f(z)}{(z - z_1)(z - z_2)(z - z_3)}$$

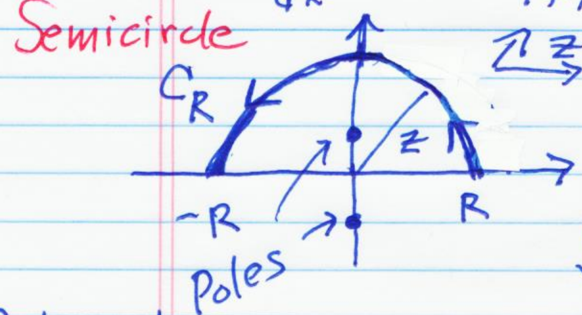
$$\oint_{\text{Any}} F(z)\,dz - 2\pi i Res(F, z_1) - 2\pi i Res(F, z_2) - 2\pi i Res(F, z_3)$$

$$\boxed{\oint_{\text{ANY}} F(z)\,dz = 2\pi i \sum_n Res(F, z_n)}$$

e.g., $Res(F, z_1) = \dfrac{f(z_1)}{(z_1 - z_2)(z_1 - z_3)}$ ← The $z - z_1$ Singularity is Cleared

T4. Complex Integration $I=\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$

$$\frac{d}{dx}\tan^{-1} = \frac{1}{1+x^2} \Rightarrow I = \tan^{-1}x\Big|_{-\infty}^{\infty} = \frac{\pi}{2}-\left(-\frac{\pi}{2}\right)=\pi$$



$$\oint \frac{dz}{1+z^2} = \int_{-\infty}^{\infty}\frac{dx}{1+x^2} + \int_{C_R}\frac{dz}{1+z^2}$$

$$\oint \frac{1}{(z+i)(z-i)}\,dz = \oint\frac{f(z)}{z-i}\,dz$$

where $f(z)=\frac{1}{z+i}$ the analytic one

But what about the semicircle?

$$I = 2\pi i\,Res(F,i) = 2\pi i\,\frac{1}{z+i}\Big|_{z=i} = 2\pi i\,\frac{1}{2i}=\pi$$

$$I_{C_R} = \int_{C_R}\frac{dz}{1+z^2} = \int_{C_R}\frac{iRe^{i\theta}\,d\theta}{1+(Re^{i\theta})^2}$$ with $z=Re^{i\theta}$, $dz = iRe^{i\theta}\,d\theta$

We want $R\to\infty$ for x-axis limits.

$$I_{C_R} = \lim_{R\to\infty}\int_{C_R}\frac{1}{1+R^2e^{2i\theta}}\,iRe^{i\theta}\,d\theta$$

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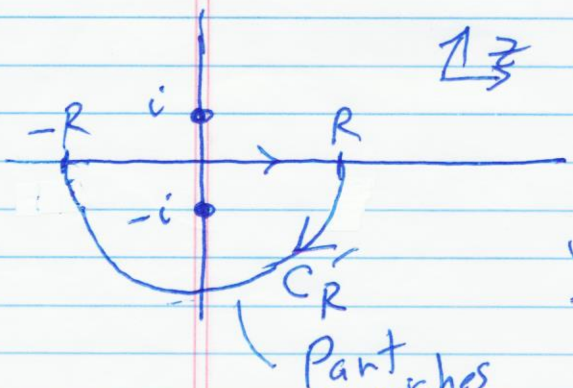
$$I_{C_R} = \lim_{R\to\infty} i\int_{C_R}\frac{e^{-2i\theta}\,e^{i\theta}}{R}\,d\theta = \underbrace{i\lim_{R\to\infty}\frac{1}{R}\int_{C_R}e^{-i\theta}\,d\theta}_{\text{zero}}$$

Close below



$$\underbrace{\oint\frac{dz}{1+z^2}}_{-2\pi i\,Res(F,-i)} = \int_{-\infty}^{\infty}\frac{dx}{1+x^2}+\int_{C_R'}\frac{dz}{1+z^2}$$

$$\left.\begin{array}{l}-2\pi i\,\underbrace{Res(F,-i)}_{\frac{1}{z-i}\big|_{-i} = -\frac{1}{2i}}\end{array}\right\}\ \pi$$

Sometimes we can only closed in either the upper or lower way. You will see later.