April 7, 2020

Class U. Green's Functions

U1. Impulse Response

Recall Radioactive Dumping. Let $n(t)$ be number left.

$$\frac{dn(t)}{dt} = \underbrace{-\lambda n(t)}_{\text{minus due to decay} \sim n(t)} + \underbrace{f(t)}_{\text{dumping (adding more)}}$$

$$f(t) = \frac{dn(t)}{dt} + \lambda n(t)$$

Take Laplace Transform

$$\mathcal{L}\{f(t)\} = \underbrace{\mathcal{L}\left\{\frac{dn(t)}{dt}\right\}}_{sN(s) - n(0)} + \lambda \underbrace{\mathcal{L}\{n(t)\}}_{N(s)}$$

$n(0)$ ∟ zero (no waste at the site when we start)

$$F(s) = sN(s) + \lambda N(s)$$

$$F(s) = N(s)[s+\lambda]$$

$$N(s) = \frac{F(s)}{s+\lambda} \Rightarrow N(s) = F(s)G(s) \qquad G(s) = \frac{1}{s+\lambda}$$

Remember: Product of Laplace Transforms $F(s)G(s) = N(s)$

$\Rightarrow n(t)$ is convolution of $f(t), g(t) \Rightarrow f(t) * g(t)$

$$n(t) = \int_0^t f(u)\, g(t-u)\, du$$

All the secrets of the system reside in $G(s)$, $g(t)$

← Isolating $G(s)$ $\quad N(s) = F(s)G(s)$

Too find the simplest $n(t)$ $\quad$ ∟ we want $F(s) = 1$

Then $N(s) = 1 \cdot G(s) \Rightarrow N(s) = G(s)$ Simplest!

What $f(t)$ has a Laplace transform of 1?

$$N(s) = F(s)G(s) \qquad \mathcal{L}\{f(t)\} = F(s) = 1$$

Remember the Sifting Property of the delta function?

$$\hookrightarrow F(s) = \int_0^\infty \underbrace{f(t)}_{\delta(t)} e^{-st} dt = 1 \qquad \text{since } e^{-st}\Big|_{t=0} = 1$$

$$\int \delta(x-a)\, h(x)\, dx = h(a)$$

$$N(s) = G(s) = \frac{1}{s+\lambda}$$

$$\int \delta(x)\, h(x)\, dx = h(0)$$

From Tables: $h(t) = g(t) = e^{-\lambda t}$

$f(t) = \delta(t)$ is an impulse

with $f(t) = A_0\, \delta(t) \Longrightarrow n(t) = g(t) = A_0\, e^{-\lambda t}$

Pristine empty site — impulse dump at $t=0$ → A_0

Radioactive Decay ↑ $e^{-\lambda t}$

George Green (1793–1841) Self-Educated!!! U-2
British Mathematical Physicist = Theoretical Physicist

U2. The Green's Function

$D[x(t)] = 0$ Differential Eq. without external input like our dumping function

$D[x(t)] = \delta(t)$ Now apply an impulse

Solution will be $g(t)$ Green's function

$$x(t) = \int_0^t f(u)\, g(t-u)\, du$$

Some arbitrary applied function like our dumping or a force in mechanics

Physics Notation

$$X(t) = \int_0^t f(u)\, G(t,u)\, du$$

$\hookrightarrow g(t-u)$ by definition for G

NoT a Laplace Transform

More Physics Notation

$$X(t) = \int_0^t G(t,u)\, f(u)\, du$$

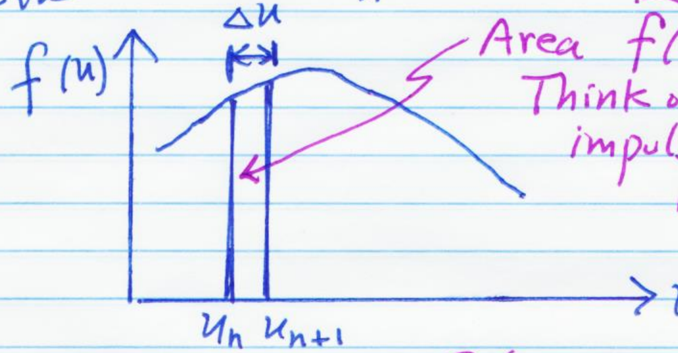
$$X(t) = \int_0^t G(t,t')\, f(t')\, dt'$$

for radioactive decay

$$G(t,t') = e^{-(t-t')}$$

Remember our disrete sum

$$X(t) = \sum_n G(t,u_n)\, \underbrace{f(u_n)\, \Delta u}$$



Area $f(u_n)\Delta u$

Think of this strip as an impulse where with hit the area with the Green's function.

$$G(t,u_n) = e^{-(t-u_n)}$$

The radioactive decay kicks in at time u_n so in the future at some time t you have $e^{-(t-u_n)} f(u_n)\Delta u$ left for that dumping at Δu

Impulse result applied to the amount dumped in Δu time.

U3. Fourier Transform Space

Recall from our Fourier Transform Class $F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$

$F(\omega) \equiv \mathcal{F}\{f(t)\}$ $\mathcal{F}\left\{\frac{df(t)}{dt}\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{df}{dt} e^{-i\omega t} dt$

Product Rule leads to integration by parts.

$$\frac{d}{dt}\left[f(t)e^{-i\omega t}\right] = \frac{df}{dt}e^{-i\omega t} - i\omega f(t) e^{-i\omega t}$$

We want this one.

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{df}{dt}e^{-i\omega t}dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{d}{dt}\left[f(t)e^{-i\omega t}\right]dt + \frac{i\omega}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

$$\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{df}{dt}e^{-i\omega t}dt}_{\mathcal{F}\left\{\frac{df}{dt}\right\}} \qquad \frac{1}{\sqrt{2\pi}} f(t) e^{-i\omega t}\Big|_{-\infty}^{\infty} + i\omega F(\omega)$$

Vanishes at $\pm\infty$ No blow-ups Boundary Conditions

$$\boxed{\mathcal{F}\left\{\frac{df}{dt}\right\} = i\omega F(\omega)} \quad \text{or } i\omega\,\mathcal{F}\{f(t)\}$$

Details here.

$\mathcal{F}\{f''\} = \mathcal{F}\{g'\}$

where $g = f'$

$\mathcal{F}\{g'\} = i\omega G(\omega)$

$\mathcal{F}\{f'\} = i\omega F(\omega)$ $\Big\}\ \underbrace{(i\omega)(i\omega)}_{-\omega^2} F(\omega)$

$$\boxed{\mathcal{F}\left\{\frac{d^2 f}{dt^2}\right\} = -\omega^2 F(\omega)}$$

U4. Finding Green's Functions

by the method extremely useful in theoretical physics

There will be 4 steps.

We will illustrate the method with the radioactive decay differential equation

Don't count this step.

Step 0. What is the differential Equation

$$\frac{dn(t)}{dt} = -\lambda n(t) + f(t)$$

$n(t)$ amount left at time t at the dumping site

$f(t)$: Dumping, external influence

$-\lambda n(t)$: radioactive decay

Step 1. Dirac Delta Function Step

$$\frac{dn(t)}{dt} = -\lambda n(t) + \delta(t)$$

Pick $f(t)$ to be an impulse

Step 2. Fourier Transform

Left Side → The System

$$\frac{d\eta(t)}{dt} + \lambda\eta(t) = \delta(t)$$

← Right Side Source from outside

$$\mathcal{F}\left\{\frac{d\eta(t)}{dt}\right\} + \lambda\,\mathcal{F}\{\eta(t)\} = \mathcal{F}\{\delta(t)\}$$

Now, we are in algebraic ω Space.

$$i\omega N(\omega) + \lambda N(\omega) = \frac{1}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{\infty}\delta(t)e^{-i\omega t}\,dt}_{1}$$

$$N(\omega)[i\omega+\lambda] = \frac{1}{\sqrt{2\pi}}$$

We need to get back to t space.

$$\boxed{N(\omega) = \frac{1}{\sqrt{2\pi}}\,\frac{1}{\lambda+i\omega}}$$

The Solution in Fourier Transform ω Space.

Step 3. Inverse Fourier Transform

$$\eta(t) = \mathcal{F}^{-1}\{N(\omega)\}$$

← Note $+i\omega t$

$$\eta(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} N(\omega)\,e^{i\omega t}\,d\omega$$

← Note ω integration

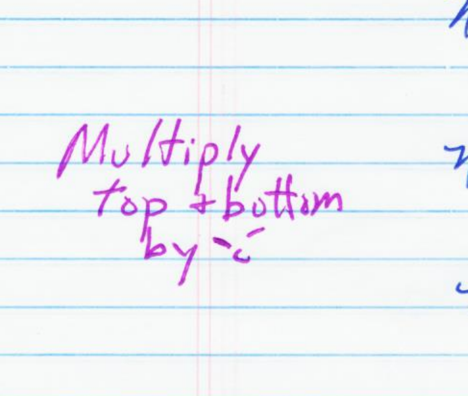
Multiply top + bottom by $-i$

$$\eta(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\,\frac{1}{\lambda+i\omega}\,e^{i\omega t}\,d\omega$$

$$\eta(t) = \frac{-i}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{\omega-\lambda i}\,d\omega$$

Step 4. Complex Integration

Promote $\omega \to z$



↳ will go to ∞

$$\underbrace{\frac{-i}{2\pi}\oint\frac{e^{izt}}{z-\lambda i}\,dz}_{\text{This one is easy to do}} = \underbrace{\frac{-i}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{\omega-\lambda i}\,d\omega}_{\eta(t)\text{ ← we want this}} - \underbrace{\frac{i}{2\pi}\int_{C_R}\frac{e^{izt}}{z-\lambda i}\,dz}_{\text{We want this to be zero}}$$

$$\underbrace{\frac{-i}{2\pi}\,2\pi i}_{+1}\,\mathrm{Res}\left(\frac{e^{izt}}{z-\lambda i}\right) = e^{izt}\Big|_{z=\lambda i} = e^{-\lambda t} = \eta(t)$$

Note: The text combines Steps 3+4.

We need to deal with

$$I_{C_R} = \frac{-i}{2\pi}\int_{C_R} \frac{e^{izt}}{z-\lambda i}\,dz \qquad \text{Let } z = Re^{i\theta}$$

$$dz = iRe^{i\theta}d\theta$$

$$0 \to \theta \to \pi$$

$$I_{C_R} = \frac{-i}{2\pi}\int_0^{\pi} \frac{e^{iRe^{i\theta}t}}{Re^{i\theta}-\lambda i}\,iRe^{i\theta}d\theta$$

$$I_{C_R} = \frac{1}{2\pi}\int_0^{\pi} \frac{e^{iR(\cos\theta + i\sin\theta)t}}{Re^{i\theta}-\lambda i}\,Re^{i\theta}d\theta$$

Note: Along the imaginary axis $z = iR$

$e^{izt} \to \underbrace{e^{-Rt} \text{ going up}}$ $\qquad e^{+Rt}$ going down the imaginary axis

Important! } Must have the semicircle above + we did that. So we are good.

$$\lim_{R\to\infty} I_{C_R} = \frac{1}{2\pi}\lim_{R\to\infty}\int_0^{\pi} \frac{e^{iR(\cos\theta)t}\,e^{-R\sin\theta t}}{1-\frac{\lambda i}{Re^{i\theta}}}\,d\theta$$

after dividing top + bottom by $Re^{i\theta}$

$e^{-R\sin\theta t} \longrightarrow 0$

Killing everything

$\frac{\lambda i}{Re^{i\theta}}$ → Will Vanish

What about $\theta = 0$ $R \to \infty$ and $e^{-R\sin\theta t}$?

The $\theta = 0$ case is no the horizontal axis and that is included in the integration from $-\infty$ to $+\infty$ along the horizontal.

So we can consider I_{C_R} starting at $\theta = 0 + \varepsilon$

└ small (vanishingly small)

So we are okay.

Step 5. The Green's Function

$$y(t) = e^{-\lambda t} = G(t,0)$$

Time Shifted Green's Function

$$G(t,t') = e^{-\lambda(t-t')}$$

General solution for $f(t)$ source function

$$y(t) = \int_0^t G(t,t') f(t')\,dt'$$

Also called a propagator in physics

The source function $f(t)$ effects propagate into the future as the Green function works on it.

Integrate over all times $t' < t$

The present.

Summary of our Journey

Use Delta Function in

Differential Equation $\xrightarrow{\text{Fourier Transform}}$ Algebraic Equation

Often the Direct Solution Not Practical

x space

t space

k space

ω space

Algebraic Techniques

Solution to the Differential Equation $\xleftarrow[\text{Transform}]{\text{Inverse Fourier}}$ Algebraic Solution

Use Complex Variable Techniques

Green's Function