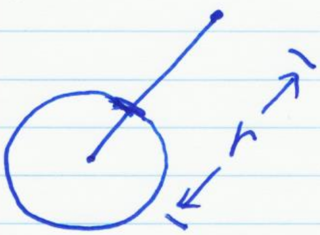
April 23, 2020

Class X. Einstein and the Precession of the Perihelion

X1. Gravitational Effect on Time $\quad E_{photon} = hf$



$$E = hf - \frac{GMm}{r} \qquad U = -\frac{GMm}{r}$$

$$hf_\infty + 0 = hf - GMm/r \quad \leftarrow E = mc^2$$

↳ at infinity $\quad$ ↳ $m = \frac{E}{c^2} = \frac{hf}{c^2}$

$$hf_\infty = hf - \frac{GM}{r}\frac{hf}{c^2}$$

Semi-Classical since $U = -\frac{GMm}{r}$ is not the General Relativity Way.

$$f_\infty = f\left(1 - \frac{GM}{c^2 r}\right)$$

$$T = T_\infty\left(1 - \frac{GM}{c^2 r}\right) \quad \text{since } T = \frac{1}{f} \quad f = \frac{1}{T}$$

$$dt' = \left(1 - \frac{GM}{c^2 r}\right)dt$$

↳ at $r = \infty$

↳ at r near the celestial body

} Turns out to be the exact GR result GR = General Relativity

What about space? The effect on Space?

X2. Gravitational Effect on Spacetime

Special relativity } SR $\quad L = L_0\sqrt{1 - v^2/c^2} \qquad T = \frac{T_0}{\sqrt{1 - v^2/c^2}}$

↑ Lorentz Contraction $\quad$ ↑ Time Dilation (stretch)

Suggests a flip $\quad dr' = \frac{dr}{\left(1 - \frac{GM}{c^2 r}\right)} \quad$ since $dt' = \left(1 - \frac{GM}{c^2 r}\right)dt$

A suggestion only. Not rigorous.

Recall: $ds^2 = c^2dt^2 - dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2 \quad$ SR

$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ in Cartesian Coordinates

$$ds^2 = \left(1 - \frac{GM}{c^2 r}\right)^2 c^2dt^2 - \frac{dr^2}{\left(1 - \frac{GM}{c^2 r}\right)^2} \quad \text{for } dt + dr$$

$\frac{GM}{c^2 r} \ll 1$ small } $\left(1 - \frac{GM}{c^2 r}\right)^2 = 1 - \frac{2GM}{c^2 r}$

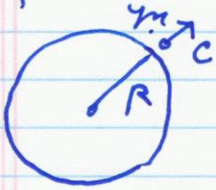
Line Element $\quad ds^2 = \left(1 - \frac{2GM}{c^2 r}\right)c^2dt^2 - \frac{dr^2}{\left(1 - \frac{2GM}{c^2 r}\right)} - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2$

↳ The Exact GR Result!

Here's another exact result. This one discovered by Michell (1795), English country parson.

To make it more known today I wrote a paper on it
M.J. Ruiz, "A Black Hole in Our Galactic Center," The Physics Teacher **47**, 10 (January 2008), featured on the cover of the journal issue.
I related the work to research of Andrea Ghez

Kick Mass m to ∞, escape the celestial body.



Escape Velocity c $\Rightarrow$ Black Hole

$$\frac{1}{2}mv^2 + \left(-\frac{GMm}{R}\right) = 0 + 0$$

mass comes to rest at ∞

$\leftarrow$ Potential Energy zero at infinity

$$\frac{1}{2}mc^2 = \frac{GMm}{R}$$

$$\frac{1}{2}c^2 = GM/R$$

Two mistakes or errors cancel.
$\frac{1}{2}mv^2$ NOT GR
$-\frac{GMm}{r}$ NOT GR
But GR $\Rightarrow$

$$\boxed{R = \frac{2GM}{c^2}}$$

Exact result from GR for the Schwarzschild radius

Pack mass inside R $\Rightarrow$ Black Hole

For Sun $M_\odot = 1.989 \times 10^{30}$ Kg One Solar Mass

$$G = 6.67 \times 10^{-11} \frac{N \cdot m^2}{Kg^2} \qquad c = 3.00 \times 10^8 \, m/s$$

$$R = \frac{2(6.67 \cdot 10^{-11})(1.989 \cdot 10^{30})}{(3.00 \cdot 10^8)^2} \text{ meters}$$

$R = 3$ Km 3 Kilometers

How can you get the mass of the Sun squeezed into 3 Km radius?

Larger stars than the Sun undergo supernova explosions. Mass gets squeezed as outer mass gets blown away

Supernova – natural mechanism.

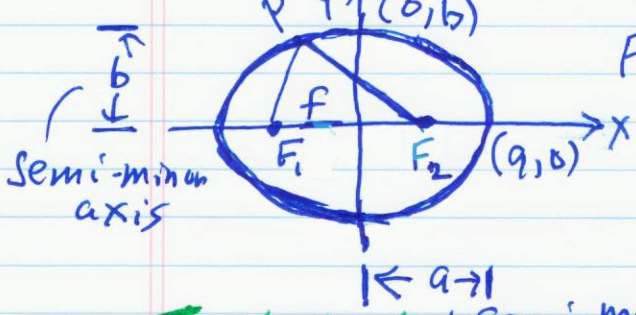
There's a black hole in the center of our galaxy.

Andrea Ghez et al find the mass is $\sim 4 \cdot 10^6 M_\odot$

4 million Solar masses

X3. Kepler's Three Laws

1st Law - Law of Ellipses - planets travel around Sun along ellipses where Sun is at a focus F_1



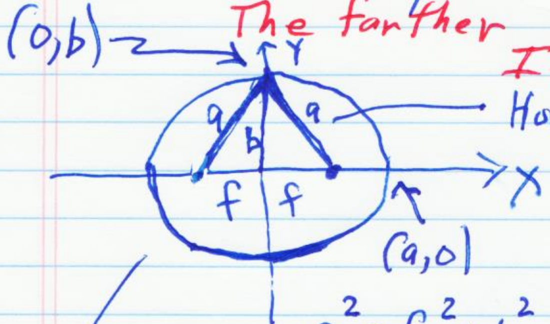
Foci F_1, F_2

Ellipse - Put thumb tacks at F_1 + F_2
Use string for place a pen at P
Where string has length F_1PF_2

Your string can have length F_1 to P to F_2 to F_1 for easy wrap around.

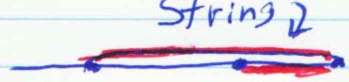
Focal distance
f is from the origin to F_1 or F_2

Eccentricity ε defined such that $f = a\varepsilon$
The farther F_1 is from the origin the more eccentric
If $\varepsilon = 0 \Rightarrow f = 0$ and we have a circle.



How is this length a?

$$F_1P + PF_2 = 2a$$



$$a^2 = f^2 + b^2 \qquad a^2 = a^2\varepsilon^2 + b^2$$

Equation of Ellipse

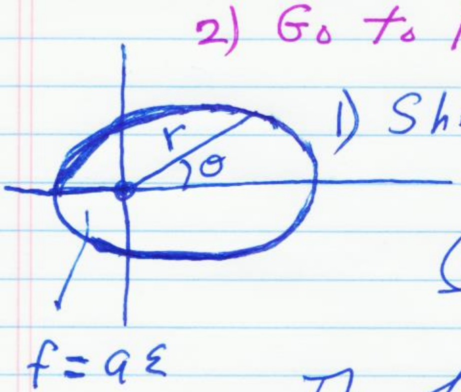
$$a^2 - b^2 = a^2\varepsilon^2 \Rightarrow 1 - \frac{b^2}{a^2} = \varepsilon^2$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\varepsilon^2 = 1 - \frac{b^2}{a^2}$$

$a = b \Rightarrow$ circle $\varepsilon = 0$
$b \to 0 \Rightarrow$ cigar $\Longrightarrow \varepsilon \to 1$
extremely elliptical

We need to do 2 things
1) Shift the ellipse so the Sun is at (0,0)
2) Go to Polar Coordinates



1) Shift by $f = a\varepsilon$ to the right $\quad x \to x - a\varepsilon$

$$\frac{(x - a\varepsilon)^2}{a^2} + \frac{y^2}{b^2} = 1$$

Then for 2) $x = r\cos\theta \quad y = r\sin\theta$

Much algebra ahead but it is high school algebra.

$$\frac{(x-a\varepsilon)^2}{a^2}+\frac{y^2}{b^2}=1 \qquad x=r\cos\theta \quad y=r\sin\theta$$

A characteristic of theoretical physics is a long derivation at times.

$$(r\cos\theta-a\varepsilon)^2/a^2+y^2/b^2=1$$

We Want $r=r(\theta)$.

$$r^2\frac{\cos^2\theta}{a^2}-2a\varepsilon\frac{r\cos\theta}{a^2}+\frac{a^2\varepsilon^2}{a^2}+\frac{r^2\sin^2\theta}{b^2}=1$$

Divide by r^2

$$\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}-\frac{2\varepsilon\cos\theta}{ra}+\frac{(\varepsilon^2-1)}{r^2}=0$$

Let $u=\frac{1}{r}$

$$-\frac{2\varepsilon\cos\theta\, u}{a} \qquad (\varepsilon^2-1)u^2$$

Multiply by $-1 \Rightarrow (1-\varepsilon^2)u^2+\frac{2\varepsilon\cos\theta}{a}u-\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)=0$

Quadratic Equation We want $u=u(\theta) \Rightarrow r=r(\theta)$.

$A=(1-\varepsilon^2)$

$B=2\varepsilon\cos\theta/a$

$C=-\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)$

$$Au^2+Bu+C=0$$

$$\frac{-B\pm\sqrt{B^2-4AC}}{2A}$$ Quadratic Formula

$$\frac{-\frac{2\varepsilon\cos\theta}{a}\pm\sqrt{\frac{4\varepsilon^2\cos^2\theta}{a^2}-4(1-\varepsilon^2)(-1)\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)}}{2(1-\varepsilon^2)}$$

Work with what's under $\sqrt{\ }$

$$\frac{4\varepsilon^2\cos^2\theta}{a^2}+4(1-\varepsilon^2)\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)$$

$$\cancel{\frac{4\varepsilon^2\cos^2\theta}{a^2}}+\frac{4\cos^2\theta}{a^2}+\frac{4\sin^2\theta}{b^2}-\cancel{\frac{4\varepsilon^2\cos^2\theta}{a^2}}-\frac{4\varepsilon^2\sin^2\theta}{b^2}$$

$$\frac{4\cos^2\theta}{a^2}+\frac{4\sin^2\theta}{b^2}-4\left(1-\frac{b^2}{a^2}\right)\frac{\sin^2\theta}{b^2} \quad \text{since } \varepsilon^2=1-\frac{b^2}{a^2}$$

$$\underbrace{\frac{4\cos^2\theta}{a^2}+\cancel{\frac{4\sin^2\theta}{b^2}}-\cancel{\frac{4\sin^2\theta}{b^2}}+\underbrace{4\frac{b^2}{a^2}\frac{\sin^2\theta}{b^2}}_{4\sin^2\theta/a^2}}_{\frac{4\cos^2\theta}{a^2}+\frac{4\sin^2\theta}{a^2}=4/a^2}$$

Nice Simplification.

$$\frac{-\frac{2\varepsilon\cos\theta}{a}\pm\sqrt{\frac{4}{a^2}}}{2(1-\varepsilon^2)}=\frac{-\frac{2\varepsilon\cos\theta}{a}\pm\frac{2}{a}}{2(1-\varepsilon^2)}$$

Solution for u

$$\frac{-\frac{2\varepsilon\cos\theta}{a} \pm \frac{2}{a}}{2(1-\varepsilon^2)} = \frac{-\varepsilon\cos\theta \pm 1}{a(1-\varepsilon^2)}$$

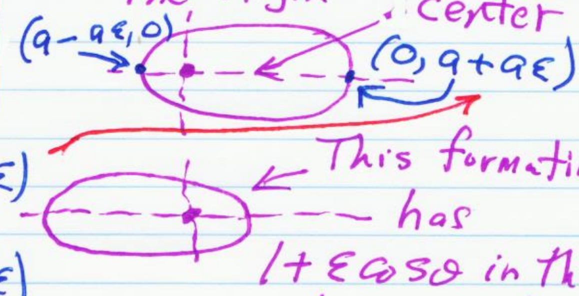
Note: $u = \frac{1}{r}$ cannot be negative.

$$u = \frac{-\varepsilon\cos\theta + 1}{a(1-\varepsilon^2)}$$

This form goes with the center of the ellipse being to the right of the origin

Law 1

$$\boxed{r = \frac{a(1-\varepsilon^2)}{1-\varepsilon\cos\theta}}$$



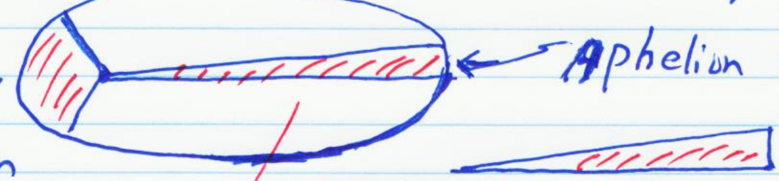
This formation has $1+\varepsilon\cos\theta$ in the denominator.

Check

$$r(\theta=0°) = \frac{a(1-\varepsilon^2)}{1-\varepsilon} = a(1+\varepsilon)$$

$$r(\theta=180°) = \frac{a(1-\varepsilon^2)}{1+\varepsilon} = a(1-\varepsilon)$$

2nd Law Law of Areas – planet sweeps out equal areas in equal times



$$dA = \tfrac{1}{2}R(Rd\theta)$$

Area of Triangle

$$dA = \tfrac{1}{2}R^2 d\theta$$

Law 2

$$\boxed{\frac{dA}{dt} = \frac{1}{2}R^2\frac{d\theta}{dt} = \text{constant}}$$

True for all Central force laws $F=F(r)$

Unlike Laws 1 + 3 which require inverse-square force.

3rd Law Law of Periods – cube of semi-major axis is proportional to square of period T

For Circular Orbit $a=R=b$

$$F=ma \Rightarrow \frac{GMm}{R^2} = m\frac{v^2}{R} = \frac{m}{R}\left(\frac{2\pi R}{T}\right)^2$$

$$\frac{GM}{R} = \frac{4\pi^2R^2}{T^2} \Rightarrow GMT^2 = 4\pi^2R^3 \Rightarrow \boxed{R^3 = \frac{GM}{4\pi^2}T^2}$$

Law 3

Poinsetta → 

Most of the 5600"/century due to gravity in Solar System

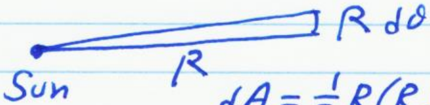
Perihelion shifts (actually entire orbit shifts — rotates)

X4. Precession of the Perihelion

* 43"/century unexplained

Reference S. Cornbleet, Am. J. Phys. 61, 650-651 (1993).

$$\frac{dA}{dt} = \frac{1}{2}R^2\frac{d\theta}{dt} = \text{const}$$



$dA = \frac{1}{2}R(R\,d\theta)$

$$dA = \underbrace{\int_0^R dr\, r\,d\theta}_{\text{Polar Coordinates}} = \frac{r^2}{2}\Big|_0^R d\theta = \frac{R^2}{2}d\theta$$

$$\text{GR} \quad ds^2 = \left(1-\frac{2GM}{c^2r}\right)c^2dt^2 - \frac{1}{\left(1-\frac{2GM}{c^2r}\right)}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2$$

Distortion in time + Space — same as classical

$$\text{GR} \quad dA' = \int_0^R dr'\,(r\,d\theta)$$

Okay here

Radial travel not okay $dr' = \frac{dr}{\sqrt{1-\frac{2GM}{c^2r}}}$

$$dr' = \left(1-\frac{2GM}{c^2r}\right)^{-\frac{1}{2}}dr$$

$$dr' \simeq \left(1+\frac{GM}{c^2r}\right)dr$$

$$dA' = \int_0^R dr'(r\,d\theta) = \int_0^R \left(1+\frac{GM}{c^2r}\right) r\,dr\,d\theta$$

$\frac{GM}{c^2r} \ll 1$ Weak gravity in Solar System. You can plug in values to verify.

$$dA' = \int_0^R \left(r + \frac{GM}{c^2}\right)dr\,d\theta$$

$$dA' = \left[\frac{r^2}{2} + \frac{GM}{c^2}r\right]_0^R d\theta$$

$$dA' = \left[\frac{R^2}{2} + \frac{GMR}{c^2}\right]d\theta$$

$$(dt')^2 = \left(1-\frac{2GM}{c^2r}\right)dt^2$$

$$dA' = \frac{R^2}{2}\left(1+\frac{2GM}{c^2R}\right)d\theta$$

$$c^2(dt')^2 = \left(1-\frac{2GM}{c^2r}\right)c^2dt^2$$

$$\frac{dA'}{dt'} = \frac{R^2}{2}\left(1+\frac{2GM}{c^2R}\right)\frac{d\theta}{dt'}$$

Distortion in Time

$$\Downarrow$$

$$\frac{d}{dt'} = \frac{1}{\sqrt{1-\frac{2GM}{c^2r}}}\frac{d}{dt} = \left(1+\frac{GM}{c^2R}\right)\frac{d}{dt}$$

$$\frac{dA'}{dt'} = \frac{R^2}{2}\left(1+\frac{2GM}{c^2R}\right)\left(1+\frac{GM}{c^2R}\right)\frac{d\theta}{dt}$$

$$\frac{dA'}{dt'} = \frac{R^2}{2}\underbrace{\left(1+\frac{2GM}{c^2R}\right)\left(1+\frac{GM}{c^2R}\right)}_{1+\frac{3GM}{c^2R} + \text{higher order in } \frac{GM}{c^2R} \ll 1}\frac{d\theta}{dt}$$

Classical $\frac{dA}{dt} = \frac{1}{2}R^2\frac{d\theta}{dt}$

General Relativity $\frac{dA'}{dt'} = \frac{R^2}{2}\left(1+\frac{3GM}{c^2R}\right)\frac{d\theta}{dt}$

Effective angle in GR of $d\theta' = \left(1+\frac{3GM}{c^2R}\right)d\theta$

$\Delta\theta \equiv \int_0^{2\pi} d\theta \leftrightarrow$ Classical $\int_0^{2\pi} d\theta = 2\pi$ What abut θ'?

$\Downarrow$

2π

$$\Delta\theta' = \int_0^{2\pi}\left(1+\frac{3GM}{c^2R}\right)d\theta$$

$$R(\theta) = \frac{a(1-\varepsilon^2)}{1-\varepsilon\cos\theta}$$

Flip

$$\frac{1}{R} = \frac{1-\varepsilon\cos\theta}{a(1-\varepsilon^2)}$$

$$\Delta\theta' = \int_0^{2\pi}\left[1+\frac{3GM}{c^2}\left(\frac{1-\varepsilon\cos\theta}{a(1-\varepsilon^2)}\right)\right]d\theta$$

$$\Delta\theta' = \underbrace{\int_0^{2\pi} d\theta}_{2\pi} + \underbrace{\frac{3GM}{c^2a(1-\varepsilon^2)}\int_0^{2\pi} d\theta}_{} - \frac{3GM}{c^2}\frac{\varepsilon}{1-\varepsilon^2}\underbrace{\int_0^{2\pi}\cos\theta\, d\theta}_{\sin\theta\big|_0^{2\pi} = 0}$$

Exact

2π Classical result $\quad \frac{3GM}{c^2a(1-\varepsilon^2)}2\pi \equiv \delta$ an advance

$\downarrow$

$G = 6.6742\times10^{-11}\,\frac{N\cdot m^2}{kg^2}$

$$\boxed{\delta = \frac{6\pi GM}{c^2a(1-\varepsilon^2)}}$$

$M_\odot = 1.98892\times10^{30}$ kg

sun

$C = 299{,}792{,}458$ m/s

X5. The 43" per Century

Mercury $a = 57.91\times10^9$ m

$\varepsilon = 0.20563$

$\Rightarrow \delta = 5.01987\times10^{-7}$ radians

Revolutions per Century $N = \frac{100 \text{ years}}{87.9391 \text{ days}} = 415.3354 \Rightarrow \Delta = N\delta = 2.08493\times10^{-4}$ rad

$\Delta = 2.08493\times10^{-4}\text{ rad}\cdot\frac{180^\circ}{\pi\text{ rad}}\cdot\frac{3600''}{1^\circ}$

$\Delta_{GR} = 43''$ /century