April 28, 2020

Class Y. Feynman's Derivation of the Schrödinger Equation

Y1. Path Integral Approach to Quantum Mechanics

Convolution for QM? Integrate over all space!

$$\psi(x,t+\varepsilon)=\int_{-\infty}^{\infty}G(x,x')\,\psi(x',t)\,dx'$$

sum over all paths.

Y2. Huygen's Principle also Huygens-Fresnel Principle

Future wave front

← Baby wavelets replace wavefront.

The Phase ϕ is $k\cdot x$.

Angle ↙ $e^{i\phi}$

Phasor $G\sim e^{ikx}$ $\quad k=\frac{2\pi}{\lambda}\quad v=\lambda f$

$$kx=\frac{2\pi}{\lambda}x=2\pi\frac{f}{v}x \quad\leftarrow\quad \frac{1}{\lambda}=\frac{f}{v}$$

But suppose the index of refraction changes.

$n\equiv\frac{c}{v}$ Index of refraction

water $n_w=1.33$

glass $n_g=1.5$

diamond $n_d=2.4$

$$\to\ kx=\underbrace{2\pi\frac{f}{c}}_{2\pi\frac{1}{\lambda_0}\text{ — vacuum}}\underbrace{\frac{c}{v}}_{\frac{c}{v}=n}x\equiv k_0 n x$$

L vacuum

$$kx\to k_0\sum_i n_i x_i$$

$$kx\to k_0\int n(x)\,dx$$

path length s often $k_0\int n(s)\,ds$

$$G\sim e^{i\phi}\qquad \phi=k_0\int n(x)\,dx$$

Minimum

Least time

$t_1=\frac{s_1}{v_1}$ $\quad n=\frac{c}{v}\quad$ $t_1=n_1s_1/c$

$t_2=\frac{s_2}{v_2}$ $\quad \frac{1}{v}=\frac{n}{c}\quad$ $t_2=n_2s_2/c$

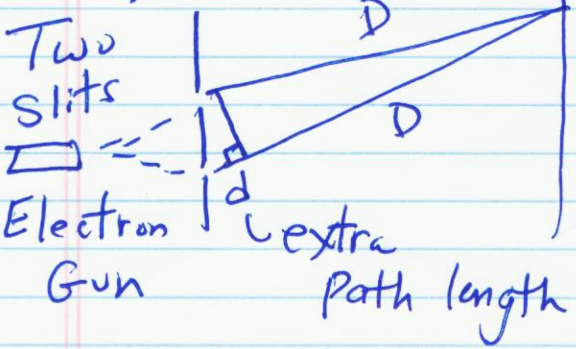
$$t=\frac{1}{c}\int n(s)\,ds$$

Fermat's Principle of Least Time

Two themes — 1) Baby waves with Phase
2) Minimum Principle ↔

Y3. Phase in QM

$\psi \sim e^{i\phi} = e^{ikx}$



Relative Phase $\Delta\phi = kd$

∟ extra path length

$k = \frac{2\pi}{\lambda} \qquad \frac{1}{\lambda} = \frac{k}{2\pi}$

de Broglie $p = \frac{h}{\lambda} \Rightarrow \frac{hk}{2\pi} \equiv \hbar k$

$$\Delta\phi = kd = \frac{\hbar k}{\hbar} d = \frac{pd}{\hbar}$$

Action $S = \int L\,dt$ for each path. Minimization Principle.

$\frac{1}{2}mv^2$ No Potential

$$S_1 = L_1 t = \frac{1}{2} m \left(\frac{D}{t}\right)^2 t = \frac{mD^2}{2t}$$

Same time

$$S_2 = L_2 t = \frac{1}{2} m \left(\frac{D+d}{t}\right)^2 t = \frac{m(D^2+2Dd+d^2)}{2t}$$

very small

$$\Delta S = S_2 - S_1 = \frac{mDd}{t} = m\left(\frac{D}{t}\right)d = \underbrace{mv}_{p}d$$

$$\left.\begin{array}{l} \Delta S = pd \\ \text{But } \Delta\phi = \frac{pd}{\hbar} \end{array}\right\} \Delta\phi = \frac{\Delta S}{\hbar}$$

Phase $\phi = k_0 \int n(s)\,ds$ $\qquad \phi = S/\hbar$

$$G \sim e^{iS/\hbar} \; ?$$

Optics ↖ $\qquad$ QM

$G \sim e^{ik_0 \int n(s)\,ds}$ $\qquad$ $G \sim e^{\frac{i}{\hbar}\int L\,dt}$

Minimize Time $\qquad$ Minimize Action

Y4. Dirac's Analogy $e^{i\phi}$

Optics	QM
$\phi = k_0 \int n(s) ds$	$\phi = \frac{1}{\hbar} \int L(t) dt$
Minimize Time	Minimize Action

units of action are same as ħ

Note that both φ are dimensionless.

$[k_0][n][s] = ?$ $\quad K_0 = \frac{2\pi}{\lambda_0} \quad \underbrace{n = \frac{c}{v}}_{\text{dimensionless}} \quad ds$

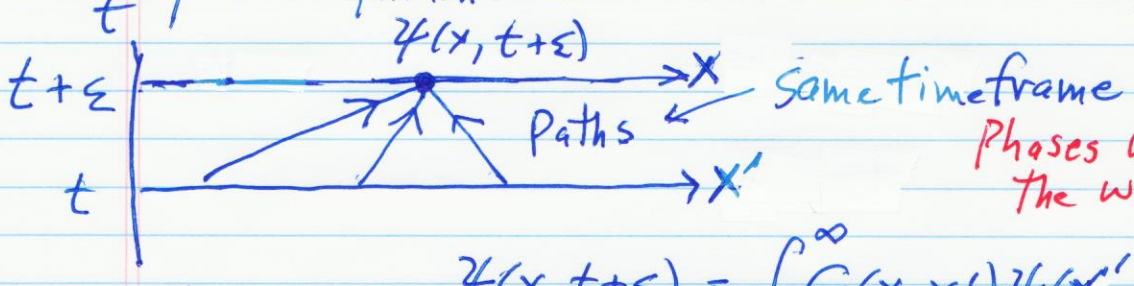
$[k_0]$: $\frac{1}{\text{meter}}$ $\quad [s]$: meter

φ radians → dimensionless (ratio of lengths)

$[L][t] \Rightarrow$ energy·time

$$[\hbar] = [h] = \underbrace{\frac{[E]}{[f]}}_{E = hf \quad h = \frac{E}{f}} = [E][T] \Rightarrow \text{energy·time}$$

Y5. Feynman's Derivation of the Schrödinger Equation



Phases will cancel for the wrong paths and you get the correct path.

$$\psi(x, t+\varepsilon) = \int_{-\infty}^{\infty} G(x,x')\psi(x',t)dx'$$

All paths from x' to x.

Kernel
Propagator } G
Green's Function

$L \quad G(x,x') = G^{iS/\hbar}$

$$S = \int_t^{t+\varepsilon} L \, dt$$

References Derbes, Am. J. Phys. 64, 881–884 (1996).
Ulul Amri, ICMSE 2015 International Conference.

$$\psi(x,t+\varepsilon) = \int G(x,x')\,\psi(x',t)\,dx' \qquad G \sim e^{iS/\hbar}$$

Equal or Proportional? Feynman tried equal at 1st

Take $G = Ae^{iS/\hbar}$

average

$$S = \int_t^{t+\varepsilon} L\,dt \approx \bar{L}\,\varepsilon \qquad \bar{L} = \bar{K} - \bar{V}$$

time interval

$$\bar{K} = \frac{1}{2}m\left(\frac{\Delta x}{\Delta t}\right)^2 = \frac{1}{2}m\frac{(x-x')^2}{\varepsilon^2}$$

go from x' to x during tim ε (all possible paths)

$$\bar{V} = V\left(\frac{x+x'}{2}\right) \qquad S = \bar{L}\varepsilon = \frac{1}{2}m\frac{(x-x')^2}{\varepsilon^2}\varepsilon - V\left(\frac{x+x'}{2}\right)\varepsilon$$

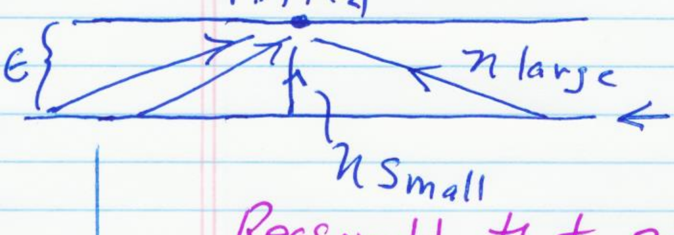
Don't forget the ε

$$G(x,x') = Ae^{iS/\hbar} \approx A\exp\left[\frac{im(x-x')^2}{2\hbar\varepsilon} - V\left(\frac{x+x'}{2}\right)\frac{i\varepsilon}{\hbar}\right]$$

$$G(x,x') = A\exp\left[\frac{im(x-x')^2}{2\hbar\varepsilon}\right]\underbrace{\exp\left[-V\left(\frac{x+x'}{2}\right)\frac{i\varepsilon}{\hbar}\right]}_{1 - V\left(\frac{x+x'}{2}\right)\frac{i\varepsilon}{\hbar}}$$

Note: that ε is very small.
The first exp has $\frac{1}{\varepsilon} \to$ large,
So we leave that alone.

Taylor Series Expansion
$e^u = 1 + u \cdots$



When ε → 0, some η will still be large.

Reasonable that path x' → x gets move contribution from nearby neighbors, i.e., $x' \approx x$.
η large – path like going a distance from New York to London will contribute less.

So let $\eta = x - x'$

$$\psi(x,t+\varepsilon) = \int_{-\infty}^{\infty} A e^{\frac{im(x-x')^2}{2\hbar\varepsilon}}\left[1 - V\left(\frac{x+x'}{2}\right)\frac{i\varepsilon}{\hbar}\right]\psi(x',t)\,dx'$$

$\eta = x - x' \quad d\eta = -dx' \qquad x' = x - \eta \quad x + x' = 2x - \eta$

$$\frac{x+x'}{2} = x - \eta/2$$

$\eta = x - x'$ $\qquad \underset{\sim}{\eta}$ $\qquad \frac{x+x'}{2} = x - \eta/2$

Y-5

$$\psi(x,t+\varepsilon) = A\int_{-\infty}^{\infty} e^{\frac{im(x-x')^2}{2\hbar\varepsilon}}\left[1 - V\left(\frac{x+x'}{2}\right)\frac{i\varepsilon}{\hbar}\right]\psi(x',t)\,dx'$$

$\eta = x - x'$ $\qquad$ └ $x-\eta$)

$-d\eta$

As $-\infty \rightarrow x' \rightarrow \infty$

$+\infty \rightarrow \eta \rightarrow -\infty$ since $d\eta = -dx'$

└ can switch back to $-\infty$-to$+\infty$ and drop the negative

$$\psi(x,t+\varepsilon) = A\int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}}\left[1 - V\left(x-\frac{\eta}{2}\right)\frac{i\varepsilon}{\hbar}\right]\psi(x-\eta,t)\,d\eta$$

Expand $\psi(x-\eta,t) = \underset{\text{0th order}}{\psi(x,t)} - \underset{\text{1st order}}{\frac{\partial\psi(x,t)}{\partial x}\eta} + \underset{\text{2nd order}}{\frac{1}{2}\frac{\partial^2\psi(x,t)}{\partial x^2}\eta^2} \ldots$

To 0^{th} order $\psi(x,t+\varepsilon) = \psi(x,t)$

$$\psi(x,t) = A\int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}}\psi(x,t)\,d\eta = \psi(x,t)\underbrace{A\int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}}d\eta}_{1}$$

Feynman first was using $A=1$ and found here that $A \neq 1$.

Recall $\int_{-\infty}^{\infty} e^{-\alpha x^2}dx = \sqrt{\frac{\pi}{\alpha}}$ $\qquad \alpha = \frac{-im}{2\hbar\varepsilon}$ $\qquad$ but there is "i" and no minus sign in the exponent!

Feynman proceeds anyway!

$$A\int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}}d\eta = A\underbrace{\sqrt{\frac{\pi}{-im/(2\hbar\varepsilon)}}}_{\sqrt{\frac{2\pi\hbar\varepsilon}{-im}}} = 1$$

must equal 1

$$A = \sqrt{\frac{-im}{2\pi\hbar\varepsilon}}$$

$$A = \sqrt{\frac{-im}{2\pi\hbar\varepsilon}\frac{i}{i}}$$

$$A = \sqrt{\frac{m}{2\pi i\hbar\varepsilon}}$$ What?

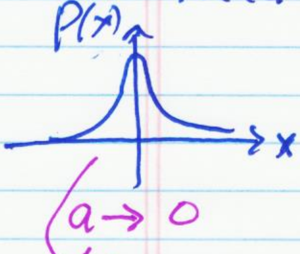
Remember for later. $A = \sqrt{\frac{\pi}{\alpha}}$

$$\psi(x,t+\varepsilon) = \sqrt{\frac{m}{2\pi i\hbar\varepsilon}}\int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}}\left[1 - V\left(x-\frac{\eta}{2}\right)\frac{i\varepsilon}{\hbar}\right]\psi(x-\eta,t)\,d\eta$$

$$\psi(x,t+\varepsilon)=\int_{-\infty}^{\infty}\underbrace{\sqrt{\frac{m}{2\pi i\hbar\varepsilon}}\,e^{\frac{im\eta^2}{2\hbar\varepsilon}}}_{\text{Focus on this combination.}}\left[1-V\left(x-\frac{\eta}{2}\right)\frac{i\varepsilon}{\hbar}\right]\psi(x-\eta,t)\,d\eta$$

Recall Gaussian in statistics $P(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}$

Let $a^2=2\sigma^2$



$P(x)\to\delta_a(x)=\frac{1}{a\sqrt{\pi}}e^{-x^2/a^2}$ Delta Sequence

$a\to0$ becomes a Spike

$\lim_{a\to0}\delta_a(x)=\delta(x)$ Dirac Delta Function

We have $\sqrt{\frac{m}{2\pi i\hbar\varepsilon}}\,e^{\frac{im\eta^2}{2\hbar\varepsilon}}$

Make the assignment $\frac{1}{a}\equiv\sqrt{\frac{m}{2i\hbar\varepsilon}}$

Then $\sqrt{\frac{m}{2\pi i\hbar\varepsilon}}=\frac{1}{a\sqrt{\pi}}$

$$\frac{1}{a^2}=\frac{m}{2i\hbar\varepsilon}=\frac{i}{i}\,\frac{m}{2i\hbar\varepsilon}=-\frac{im}{2\hbar\varepsilon}$$

$$-\frac{1}{a^2}=+\frac{im}{2\hbar\varepsilon}$$

So $\sqrt{\frac{m}{2\pi i\hbar\varepsilon}}\,e^{im\eta^2/2\hbar\varepsilon}\to\frac{1}{a\sqrt{\pi}}e^{-\eta^2/a^2}$

Feynman and Wizardry! This entire derivation.

What?

$$\delta_\varepsilon(\eta)=\sqrt{\frac{m}{2\pi i\hbar\varepsilon}}\,e^{\frac{im\eta^2}{2\hbar\varepsilon}}$$

$$\psi(x,t+\varepsilon)\simeq\int_{-\infty}^{\infty}\underbrace{\delta_\varepsilon(\eta)}_{\delta(\eta)\text{ as }\varepsilon\to0}\left[1-V\left(x-\frac{\eta}{2}\right)\frac{i\varepsilon}{\hbar}\right]\psi(x-\eta,t)\,d\eta$$

Only neighbors weigh in, as we expected.

Justifies expansion in η.

Two small quantities

$$V(x-\tfrac{\eta}{2}) = V(x) - V'(x)\frac{\eta}{2}$$

$$V(x-\tfrac{\eta}{2})\varepsilon = V(x)\varepsilon - V'(x)\frac{\eta}{2}\varepsilon \approx V(x)\varepsilon$$

$$\psi(x,t+\varepsilon) \simeq A\int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}}\left[1-V(x)\frac{i\varepsilon}{\hbar}\right]\psi(x-\eta,t)\,d\eta$$

$$A = \sqrt{\frac{m}{2\pi i\hbar\varepsilon}}$$

$$\psi(x-\eta,t) = \psi(x,t) - \frac{\partial\psi(x,t)}{\partial x}\eta + \frac{1}{2}\frac{\partial^2\psi(x,t)}{\partial x^2}\eta^2 \cdots$$

Three Integrals ①, ②, ③

$$I_1 = \psi(x,t)\,A\int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}}\left[1-V(x)\frac{i\varepsilon}{\hbar}\right]d\eta$$ To 1st order in ε

$$I_2 = -\frac{\partial\psi(x,t)}{\partial x}A\int_{-\infty}^{\infty} e^{\overset{\text{even}}{\frac{im\eta^2}{2\hbar\varepsilon}}}\overset{\text{odd}}{\eta}\,d\eta$$, neglecting the $\varepsilon\eta$ term

$I_2 = 0$

$$I_3 = \frac{1}{2}\frac{\partial^2\psi(x,t)}{\partial x^2}A\int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}}\eta^2\,d\eta$$, neglecting the $\varepsilon\eta^2$ term

→ Why keep this η^2 term if we have the η term I_2? Because $I_2 = 0$

Note e^{iu^2} is even $\cos u^2 + i\sin^2 u$

$$I_1 = \psi(x,t)\left[1-V(x)\frac{i\varepsilon}{\hbar}\right]\underbrace{A\int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}}d\eta}_{1}$$

$I_2 = 0$

$$I_3 = \frac{1}{2}\frac{\partial^2\psi}{\partial x^2}\underbrace{A\int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}}\eta^2\,d\eta}$$

Feynman was fond of the derivative trick.

$A = \sqrt{\frac{m}{2\pi i\varepsilon\hbar}}$

$\alpha = \frac{-im}{2\hbar\varepsilon}$

$$\underbrace{\int_{-\infty}^{\infty} e^{-\alpha\eta^2}\eta^2\,d\eta}_{\frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}} = -\frac{d}{d\alpha}\underbrace{\int_{-\infty}^{\infty} e^{-\alpha\eta^2}dx}_{\sqrt{\pi/\alpha}}$$

←

$$\psi(x,t+\varepsilon) = I_1 + \cancelto{0}{I_2} + I_3$$

$$\llcorner\ I_1 = \psi(x,t)\left[1 - V(x)\,\frac{i\varepsilon}{\hbar}\right]$$

$$I_3 = \frac{1}{2}\frac{\partial^2\psi}{\partial x^2} A \underbrace{\int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}}\,\eta^2\,d\eta} \qquad \alpha = -\frac{im}{2\hbar\varepsilon}$$

$$\text{Note: } \underbrace{A = \sqrt{\frac{\alpha}{\pi}}}_{\text{Normalization from earlier}} \qquad \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}} = \frac{1}{2}\left(\frac{-2\hbar\varepsilon}{im}\right)\sqrt{\frac{\pi}{\alpha}} \searrow$$

$$A = \sqrt{\frac{m}{2\pi i\varepsilon\hbar}}$$

$$I_3 = \frac{1}{2}\frac{\partial^2\psi}{\partial x^2} \underbrace{A}_{\hookrightarrow \sqrt{\frac{\alpha}{\pi}}} \frac{1}{2}\left(-\frac{2\hbar\varepsilon}{im}\right)\sqrt{\frac{\pi}{\alpha}} \nearrow \underbrace{\frac{1}{2}\left(-\frac{2\hbar\varepsilon}{im}\right)}_{-\frac{\hbar\varepsilon}{im} = \frac{i\hbar\varepsilon}{m}}$$

$$I_3 = \frac{i\hbar\varepsilon}{2m}\frac{\partial^2\psi}{\partial x^2}$$

$$\psi(x,t+\varepsilon) = \underbrace{\psi(x,t)\left[1 - V(x)\frac{i\varepsilon}{\hbar}\right]}_{\psi(x,t) - \psi(x,t)V(x)\frac{i\varepsilon}{\hbar}} + \frac{i\hbar\varepsilon}{2m}\frac{\partial^2\psi}{\partial x^2}$$

$$\psi(x,t+\varepsilon) - \psi(x,t) = -\psi(x,t)V(x)\frac{i\varepsilon}{\hbar} + \frac{i\hbar\varepsilon}{2m}\frac{\partial^2\psi}{\partial x^2}$$

$$\underbrace{\frac{\psi(x,t+\varepsilon) - \psi(x,t)}{\varepsilon}}_{\frac{\partial\psi(x,t)}{\partial t}} = -\psi(x,t)V(x)\frac{i}{\hbar} + \frac{i\hbar}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}$$

Mult. by $i\hbar$

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = \psi(x,t)V(x) - \frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}$$

Schrödinger Equation

$$\boxed{i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t)}$$

Note the "i" and $\hbar^2/2m$

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi$$

Gives the evolution of the state!