Theoretical Physics
Prof. Ruiz, UNC Asheville
Chapter A Homework. Taylor Series, Rotation Matrix, Groups

HW-A1. Taylor Series. Find the Maclaurin series, i.e., Taylor series about $x = 0$, up to cubic power in x for the function $f(x) = \cos(x)e^x$.

a) Method 1. Use the traditional method taking derivatives.

b) Method 2. Check your calculation by multiplying the Maclaurin series for $\cos(x)$ up to cubic power in x by the Maclaurin series for e^x up to up to cubic power in x. When you do this multiplication, you will get higher powers than x^3. But since we are only interested in the terms up to and including x^3, you can toss the higher powers away. Such an approximation is often done in physics for small $x << 1$, where we frequently neglect higher-order terms. Sometimes we just keep the first power or second. In this problem, you are keeping up to power x^3. You will need the following formulas for Part (b), which by the way are good to have memorized

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \quad \text{and} \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\ldots$$

HW-A2. Matrix Multiplication. Consider the following three matrices, called the Pauli matrices.

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

a) Calculate the matrix product $\sigma_x \sigma_y$ and express your answer in terms of σ_x, σ_y, and/or σ_z.

b) Calculate the matrix product $\sigma_y \sigma_x$, i.e., in reverse order. Again, express your answer in terms of σ_x, σ_y, and/or σ_z.

c) What is the commutator defined as $[\sigma_x, \sigma_y] \equiv \sigma_x \sigma_y - \sigma_y \sigma_x$?

d) What is the anticommutator defined as $\{\sigma_x, \sigma_y\} \equiv \sigma_x \sigma_y + \sigma_y \sigma_x$?

HW-A3. Rotation Matrix. Show that

$$R(30°) = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, \quad R(60°) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \text{ and } R(90°) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then by explicit matrix multiplications, show that

$$R(90°) = R(30°)R(60°) = R(60°)R(30°) .$$

HW-A4. A Three-Element Group. The set $S = \{R(0°), R(120°), R(-120°)\}$ consists of three elements where R stands for a rotation. A binary operation is defined where $R(\alpha) \cdot R(\beta)$ means you apply one rotation after the other.

a) Closure. Show that closure is met by constructing the complete multiplication table for the group. Your multiplication table will have 9 entries where each of these entries will be one of the three elements from the set: $R(0°)$, $R(120°)$, or $R(-120°)$.

b) Association. Demonstrate that association is satisfied.

c) Identity. Show that there is an identity element.

d) Inverse. Show that every element in the set has an inverse in the same set.

Is the group Abelian? Why or why not?