Theoretical Physics
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Chapter B Notes. What is e? Euler's Formula, Integral Tricks

B1. What is e?

We can define e such that $\dfrac{de^x}{dx} = e^x$, i.e., the derivative is itself. Then, the power series that makes this possible is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots .$$

Take a derivative and check it out. You encounter this natural base in radioactive decay and exponential growth problems. The number "e" is also called Euler's number. In such problems the time rate of chance of the number of particles is proportional to the number present. In other words, the derivative is proportional to the same function.

But also consider the binomial expansion

$$(a+b)^n = a^n b^0 + \frac{n}{1} a^{n-1} b^1 + \frac{n(n-1)}{1 \cdot 2} a^{n-2} b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3} b^3 + \dots$$

and the specific case below:

$$(1+\frac{x}{n})^n = 1 + \frac{n}{1}\left[\frac{x}{n}\right] + \frac{n(n-1)}{1 \cdot 2}\left[\frac{x}{n}\right]^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}\left[\frac{x}{n}\right]^3 + \dots$$

$$\lim_{n \to \infty}(1+\frac{x}{n})^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = e^x$$

PB1 (Practice Problem). Why does the above work out like that? Show the detailed steps for the first four terms. What happens to the nth term?

B2. Economics and e

Next, we turn to economics for more insight into this natural base. The compound interest formula is related to e. Take x = 1, i.e., 100% interest, and consider interest applied once per year, twice per year, three times, and so on.

$$(1+1)^1 = 1+1 = 2$$

$$(1+\frac{1}{2})(1+\frac{1}{2}) = (1+\frac{1}{2})^2 = 1+1+\frac{1}{4} = 2.25$$

$$(1+\frac{1}{3})^3 = 1+3\left[\frac{1}{3}\right]+3\left[\frac{1}{3}\right]^2+\left[\frac{1}{3}\right]^3 = 1+1+\frac{1}{3}+\frac{1}{27} = 2.37$$

$$(1+\frac{1}{4})^4 = 1+4\left[\frac{1}{4}\right]+6\left[\frac{1}{4}\right]^2+4\left[\frac{1}{4}\right]^3+\left[\frac{1}{4}\right]^4 = 1+1+\frac{3}{8}+\frac{1}{16}+\frac{1}{256} = 2.44$$

The instantaneous compounding of interest at 100% gives \$2.72, i.e.,

$$e^x = \lim_{n\to\infty}(1+\frac{x}{n})^n \quad \text{and} \quad e = \lim_{n\to\infty}(1+\frac{1}{n})^n = 2.71828\ldots$$

$$e = 2.71828\ 18284\ 59045\ 23536\ 02874\ \ldots$$

There is no regularly repeating pattern. Therefore, e is said to be ***irrational***, i.e., it cannot be expressed as a ratio of two integers. Another example of an irrational number is

$$\pi = 3.14159\ 26535\ 89793\ 23846\ 26433\ \ldots$$

Furthermore, both e and π cannot be obtained as solutions to any integer polynomial equation. Therefore, these numbers are also said to be ***transcendental***.

Can you think of a number that is irrational but not transcendental? One answer is

$$\sqrt{2}\ .$$

This number is not transcendental because it can be expressed as a solution to an integer polynomial equation such as

$$x^2 = 2\ .$$

You can't get e or π in such a way. And you can't cheat with $x^2 = \pi^2$ or $x-\pi=0$ since then you are NOT using only integers.

B3. This is Our Jewel: Euler's Formula

Leonhard Euler (1707-1783)
Courtesy School of Mathematics and Statistics
University of St. Andrews, Scotland

Five important numbers:

0 (the additive identity element for the group of integers under addition Z)

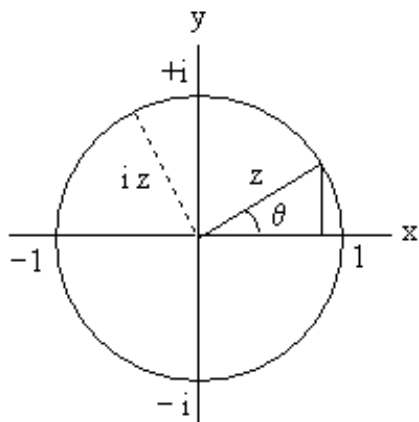1 (the multiplicative identity element for the group of nonzero rational numbers Q)

i (the imaginary number)

π (ratio of circumference to diameter of a circle)

e (the natural base)

We will give a formula that relates all five of these shortly.

Below is a unit circle where z = cos θ + i sin θ. See below to understand why dz/dθ = iz. Note that the derivative is in the tangential direction, i.e., 90° with respect to the z-line in the complex plane. Remember "Left Face" from our discussion of groups. Since dz/dθ = iz, then z = e$^{i\theta}$ = cos θ + i sin θ. This is the Euler formula.

$$z = \cos\theta + i\sin\theta$$

z rotated by 90°

$$dz/d\theta = -\sin\theta + i\cos\theta = iz$$

$$z = e^{i\theta}$$

PB2 (Practice Problem). Take z = x + iy and then show that iz = ix − y is indeed the rotated version of x + iy by 90 degrees from analyzing the legs of the right triangles formed by z and iz.

In summary, the Euler formula is

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Physicist Richard Feynman, in his famous lectures on physics delivered at Caltech and in book form, states that the Euler Formula is the most remarkable formula in mathematics! He states "This is our jewel."
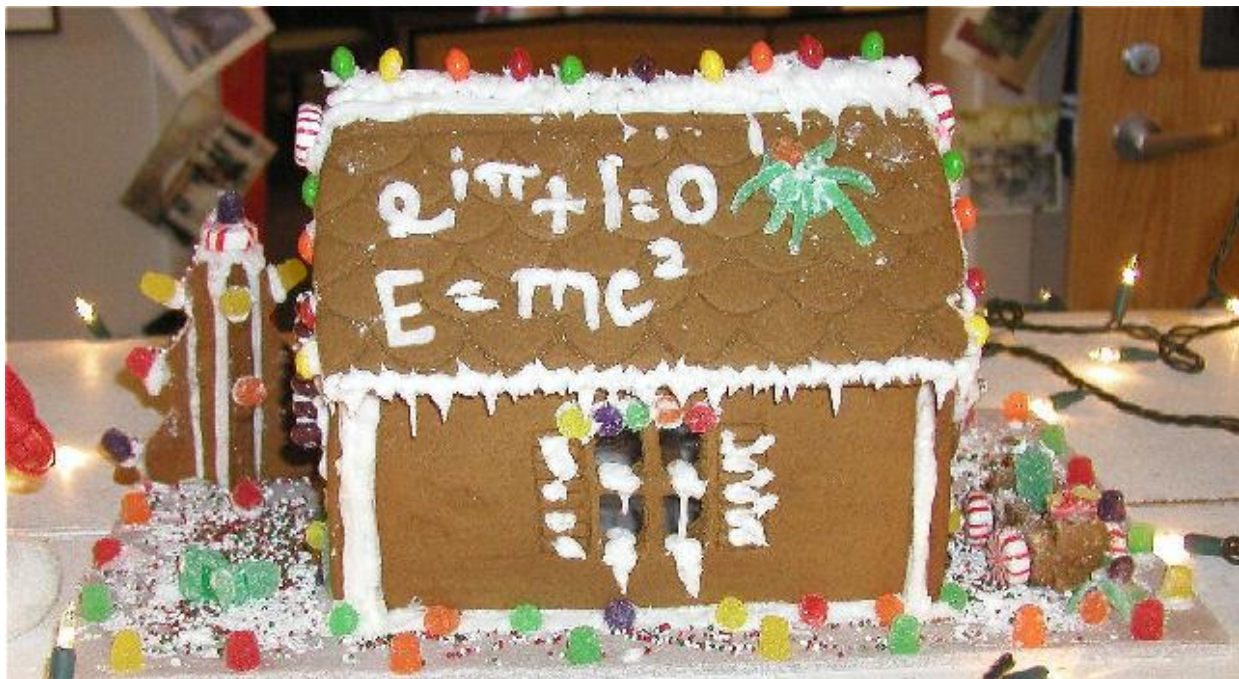
B4. This is Our Opal: Euler's Identity

So, the secret to connecting i, π and e is to study the unit circle in the complex plane. Let $\theta = \pi$ in the Euler Formula $e^{i\theta} = \cos\theta + i\sin\theta$:

$$e^{i\pi} = -1.$$

This is equivalent to the following magic formula, called *Euler's Identity*.

$$e^{i\pi} + 1 = 0$$

The Rhoades-Robinson Gingerbread House (2004)

To see the full Gingerbread Album, go to:

http://www.mjtruiz.com/gingerbread.php

You might enjoy reading the explanations I prepared there for the smart high school student.

I would like to share with you another derivation of this marvelous formula. Let's play around with the circle in the complex plane to see what further insight we can gain. The main challenges here are two: 1) relate i to π and then 2) bring "e" into the game somehow.

The first we already did because multiplying by i is a rotation. Rotation is easily connected to a circle. So the rotation by 90° or $\pi/2$ radians means we have established a connection between i and π.

But now we are very close in bringing e into this. If you multiply by i two times you get to minus 1. But this is 180° or π. If you multiply by i a third time, i.e., i cubed, you add another $\pi/2$ radians. Hey, we are multiplying here and adding something else that goes with it. This means logarithms and logarithms mean exponentials.
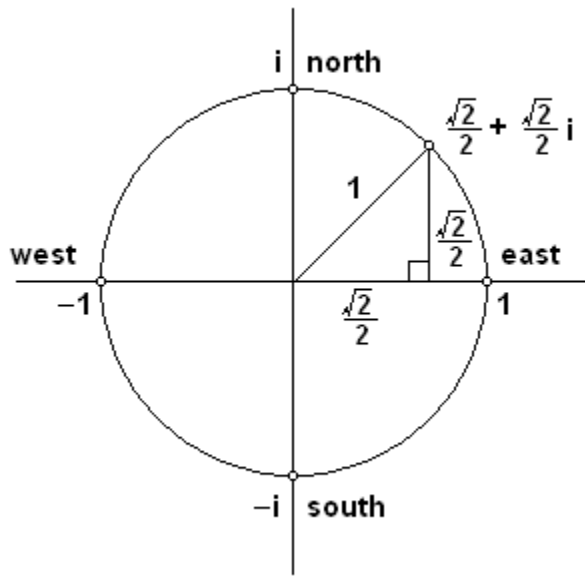

If you square i (north) you get to -1 (west). We can go backwards by taking the square root. The square root of minus one (west) is found by rotating 90° (north) instead of the 180° for the minus one (west). You halve the angle. Does this mean 90°/2 = 45° corresponds the square root of i?

The northeast number, i.e., the number at 45° is $\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2} i = \dfrac{\sqrt{2}}{2}(1+i)$. This is crazy looking. Can squaring this northeast number really take us to north, i.e., i. You bet.

$$\left[\frac{\sqrt{2}}{2}(1+i)\right]^2 = \frac{1}{2}(1+i)^2 = \frac{1}{2}(1+2i+i^2) = \frac{1}{2}(1+2i-1) = i$$

So $\sqrt{i} = \dfrac{\sqrt{2}}{2}(1+i)$. What about the fourth root of i? It's got to be 22.5°.

PB3 (Practice Problem). What is the square root of minus i? Check your answer by working it out.
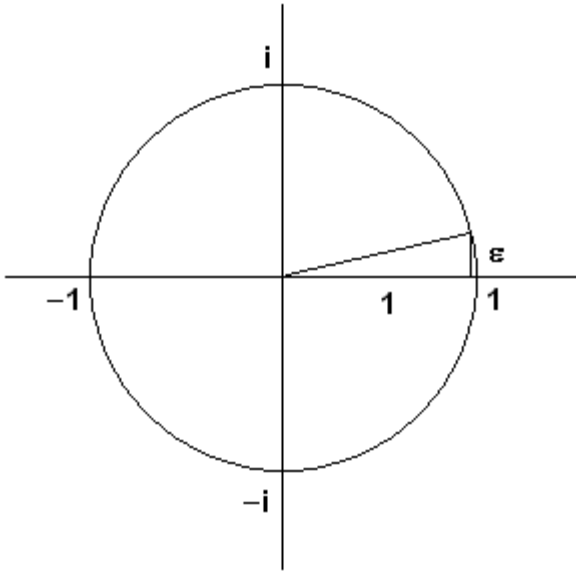
I am going to start with $z = 1 + i\varepsilon$. If I multiply this by itself, I double the angle. Let's make the angle we start with to be 1/n of the journey to -1, which resides at pi, and pick n to be large. Then the little vertical line gets closer and closer in size to the arc length as the angle gets smaller and smaller.

We then can write $\varepsilon = \dfrac{\pi}{n}$ and

$$-1 = \lim_{n \to \infty}(1 + \frac{i\pi}{n})^n$$

But from our earlier result that

$$\lim_{n \to \infty}(1 + \frac{x}{n})^n = e^x$$

we find

$$-1 = \lim_{n \to \infty}(1 + \frac{i\pi}{n})^n = e^{i\pi},$$

which can be rewritten as

$$e^{i\pi} + 1 = 0.$$

B5. Integral Trick 1: **The Real-Imaginary Trick**

How would you go about integrating the following where $a > 0$?

$$\int_0^\infty e^{-ax} \cos(kx) dx$$

Would you believe it is easier to do both the cos(kx) and sin(kx) integrals at the same time. The imaginary number "i" can keep the integrations from contaminating each other.

$$\int_0^\infty e^{-ax} \left[\cos(kx) + i\sin(kx)\right] dx = \int_0^\infty e^{-ax} e^{ikx} dx = \left.\frac{e^{-ax+ikx}}{-a+ik}\right|_0^\infty = \frac{0-1}{-a+ik}$$

$$= \frac{1}{a-ik} = \frac{1}{(a-ik)}\left[\frac{a+ik}{a+ik}\right] = \frac{a+ik}{a^2+k^2}$$

By comparing real and imaginary components, we have our two integrals.

$$\int_0^\infty e^{-ax} \cos(kx) \, dx = \frac{a}{a^2+k^2}$$

$$\int_0^\infty e^{-ax} \sin(kx) \, dx = \frac{k}{a^2+k^2}$$

B6. Integral Trick 2: The Polar Coordinates Trick

When I was a freshman, my best friend told me his calculus teacher sent the class home one day to try to evaluate the challenging integral below. The teacher was being tricky. He knew the students couldn't use the simple tricks they had learned so far.

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx = ?$$

I tried at the time, but I could not help my friend. My friend asked his older brother who was then a first-year graduate student in physics. The reply was to use a Taylor series expansion, but he didn't give any more details. About five years later at the *University of Maryland*, a friend of mine and graduate from *Caltech* explained the trick to me when he was driving me in a car one day. You might want to challenge yourself to re-derive the result in your head later, but not while you are driving.

Would you believe it is easier to evaluate the square of this integral? Then you take the square root of your result to get the final answer. You must be careful and introduce y for the second expression of the integral since the integrations are distinct from each other.

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy = \int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} e^{-r^2}\, r dr d\theta$$

You see the real trick here is our switching to polar coordinates.

$$I^2 = \int_{\theta=0}^{2\pi} d\theta \int_{r=0}^{\infty} e^{-r^2}\, r dr = 2\pi \frac{e^{-r^2}}{-2}\bigg|_{0}^{\infty} = 2\pi\left[0 - \left(\frac{1}{-2}\right)\right] = \pi$$

Our integration covers the entire two-dimensional plane. The answer for our original integral is

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}\ .$$

We thus arrive at another mysterious connection between e and π.

PB4 (Practice Problem). What is the value for the following integral, where $a > 0$?

$$I = \int_{-\infty}^{\infty} e^{-ax^2}\, dx = ?$$

B7. Integral Trick 3: **The Derivative Trick**

What about this integral?

$$I = \int_{-\infty}^{\infty} x^2 e^{-x^2} \, dx = ?$$

In the last section you found

$$I = \int_{-\infty}^{\infty} e^{-ax^2} \, dx = \sqrt{\frac{\pi}{a}} \, ,$$

where $a > 0$. We are going to do our integral by taking a derivative and it is understood that we will set a to 1 after we are finished.

$$I = \int_{-\infty}^{\infty} x^2 e^{-x^2} \, dx = -\frac{d}{da} \int_{-\infty}^{\infty} e^{-ax^2} \, dx = -\frac{d}{da} \sqrt{\frac{\pi}{a}} = \frac{1}{2a} \sqrt{\frac{\pi}{a}}$$

Do you know anyone who would rather integrate than differentiate? We can now set a = 1 to obtain the result for our original integral.

$$I = \int_{-\infty}^{\infty} x^2 e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}$$

PB5 (Practice Problem). Find the general formula for

$$I = \int_{-\infty}^{\infty} x^{2n} e^{-x^2} \, dx$$

by taking derivatives over and over again.

B8. Integral Trick 4: **Completing the Square**

Here is a scary integral:

$$I = \int_{-\infty}^{\infty} e^{-ax^2+ikx} \, dx$$

,

where $a > 0$. We do this integral by completing the square in the exponent. Consider

$$ax^2 - bx, \text{ where } a > 0$$

and take b = ik. You complete the square like this

$$ax^2 - bx = a(x - \frac{b}{2a})^2 - \frac{b^2}{4a}.$$

Then,

$$\int_{-\infty}^{\infty} e^{-ax^2+bx} \, dx = \int_{-\infty}^{\infty} e^{-a(x-\frac{b}{2a})^2} e^{\frac{b^2}{4a}} dx = e^{\frac{b^2}{4a}} \int_{-\infty}^{\infty} e^{-az^2} \, dz = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}$$

Setting b = ik gives

$$\int_{-\infty}^{\infty} e^{-ax^2+ikx} \, dx = \sqrt{\frac{\pi}{a}} e^{-\frac{k^2}{4a}}$$

,

where of course $a > 0$.