**Theoretical Physics**
**Prof. Ruiz, UNC Asheville**
**Chapter C Solutions. Relativity and Four Vectors**

**HW-C1. Time Dilation in GPS Satellites.** Find the time that a GPS satellite clock in a 12-hour orbit around the Earth loses in microseconds each day.

a) The radius of the orbit to four significant figures. Use Kepler's Third Law

$$r^3 = \frac{GM}{4\pi^2}T^2 = \frac{(6.6743\times10^{-11}\ \frac{\mathrm{N}\times\mathrm{m}^2}{\mathrm{kg}^2})(5.9722\times10^{24}\,\mathrm{kg})}{4\pi^2}(12\mathrm{h}\cdot3600\,\frac{\mathrm{s}}{\mathrm{h}})^2$$

$$r^3 = 1.88429\times10^{22}\ \mathrm{m}^3 \quad => \quad r = 2.66103\times10^7\ \mathrm{m}$$

$$\boxed{r = 2.661\times10^7\ \mathrm{m}}$$

b) Orbital speed to four significant figures.

$$v = \frac{2\pi r}{T} = \frac{2\pi(2.66103\times10^7\ \mathrm{m})}{12\mathrm{h}\cdot3600\,\frac{\mathrm{s}}{\mathrm{h}}} = 3870.3\ \frac{\mathrm{m}}{s}$$

$$\boxed{v = 3870.\ \frac{\mathrm{m}}{s}}$$

Note the decimal point to indicate that the zero in this case is significant.

c) Time lost per day in microseconds to 3 significant figures. We kept 4 significant figures above to avoid rounding until the end. The speed of light is exactly 299,792,458 m/s.

$$T_{Satellite} = T_{Earth}\sqrt{1-\frac{v^2}{c^2}}$$

Since v << c, we can do a Taylor series expansion here using $(1+x)^n \approx 1+nx$.

$$T_{Satellite} = T_{Earth}(1-\frac{1}{2}\frac{v^2}{c^2})$$

Are you worried about significant figures? The next term will be the $\frac{v^4}{c^4}$ term. Since

$$\frac{v^2}{c^2} = (\frac{v}{c})^2 = (\frac{3870}{300,000,000})^2 = 1.664 \times 10^{-10},$$

the $\frac{v^4}{c^4}$ term is on the order of $10^{-20}$, way out there by 10 decimal places.

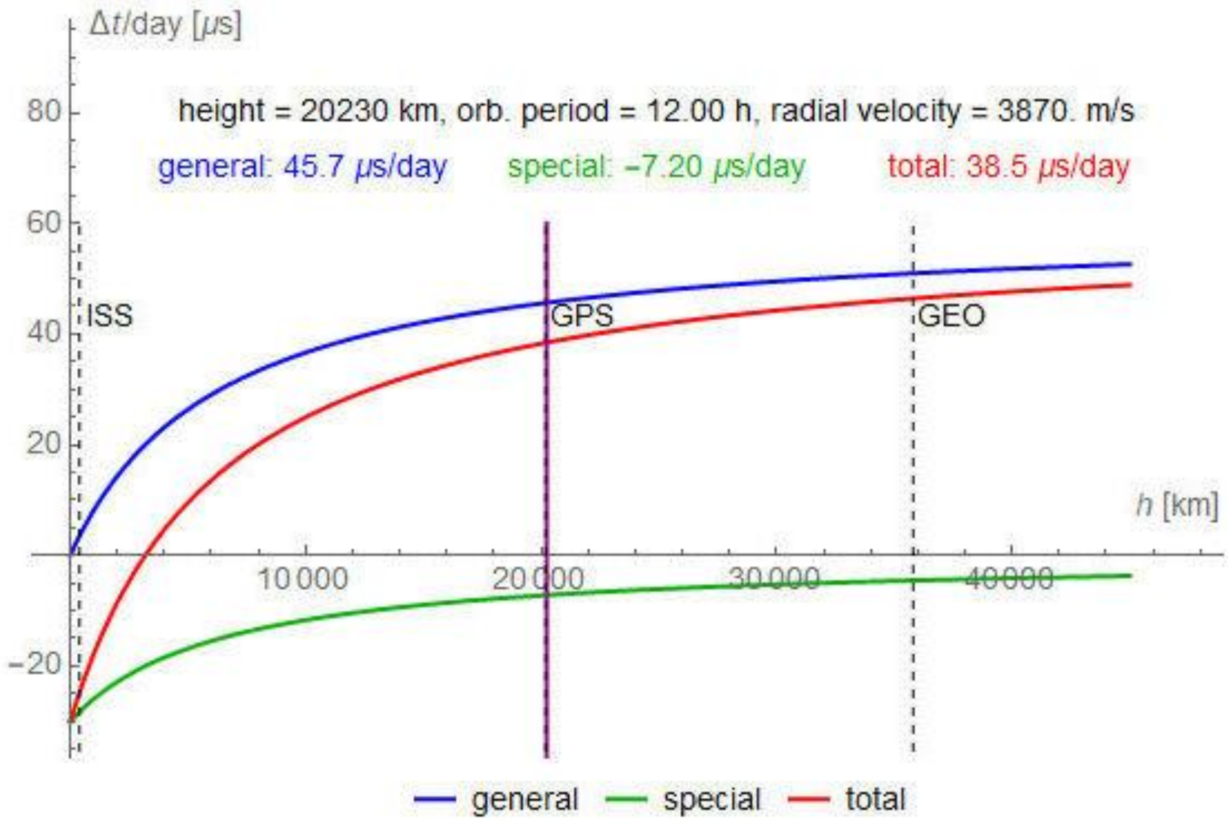
$T_{Satellite} = T_{Earth}(1 - \frac{1}{2}\frac{v^2}{c^2})$. The time lost is $\Delta T_{Satellite} = -T_{Earth}\frac{1}{2}\frac{v^2}{c^2}$,

where below we keep the minus sign (optional) to remind us that it is time lost.

$$\Delta T_{Satellite} = -T_{Earth}\frac{1}{2}\frac{v^2}{c^2} = -24\ \text{h} \cdot \frac{3600\ \text{s}}{\text{h}} \frac{1}{2}(\frac{3870}{299,792,458})^2$$

$$\Delta T_{Satellite} = -7.20 \times 10^{-6}\ \text{s} \quad => \quad \boxed{\Delta T_{Satellite} = -7.20\ \mu\text{s/day}}$$

It would be nice to check this result since we did many steps. WOLFRAM has an app.



**HW-C2. Perpendicular Velocity Formula.** Derive the relativistic formula for the perpendicular velocity transformation where the object moves in the K' frame with speed $\vec{u}' = (u_x', u_y')$ and the K frame measures $\vec{u} = (u_x, u_y)$. The K' frame moves at the usual speed $v$ in the x-direction relative to the K frame. Your answer will be

$$u_y = \frac{u_y'\sqrt{1-\frac{v^2}{c^2}}}{1+\frac{u_x'v}{c^2}}.$$

**Solution (Start with x and t formulas from the book).**

$$x = \frac{x'+vt'}{\sqrt{1-\frac{v^2}{c^2}}}, \quad t = \frac{t'+x'\frac{v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$, and $y = y'$ since K' moves in the x direction.

$$\Delta x = \frac{\Delta x'+v\Delta t'}{\sqrt{1-\frac{v^2}{c^2}}}, \quad \Delta t = \frac{\Delta t'+\Delta x'\frac{v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$, and $\Delta y = \Delta y'$

$$\frac{\Delta y}{\Delta t} = \Delta y' \div \frac{\Delta t'+\Delta x'\frac{v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\Delta y'\sqrt{1-\frac{v^2}{c^2}}}{\Delta t'+\Delta x'\frac{v}{c^2}}$$

Dividing top and bottom of the right side of $\frac{\Delta y}{\Delta t} = \frac{\Delta y'\sqrt{1-\frac{v^2}{c^2}}}{\Delta t'+\Delta x'\frac{v}{c^2}}$ by $\Delta t'$. We get

$$\frac{\Delta y}{\Delta t} = \frac{\frac{\Delta y'}{\Delta t'}\sqrt{1-\frac{v^2}{c^2}}}{1+\frac{\Delta x'}{\Delta t'}\frac{v}{c^2}}$$, which gives $\boxed{u_y = \frac{u_y'\sqrt{1-\frac{v^2}{c^2}}}{1+\frac{u_x'v}{c^2}}}$.