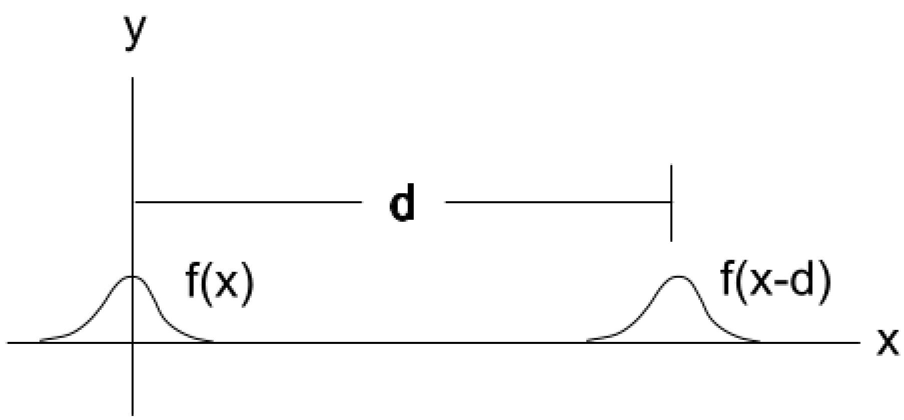
Theoretical Physics
Prof. Ruiz, UNC Asheville, doctorphys on YouTube
Chapter F Notes. "Let There Be Light"

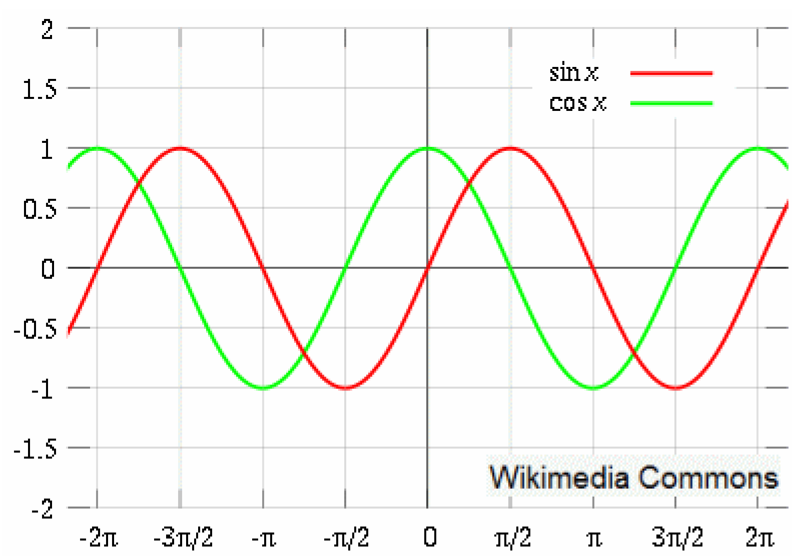
F1. The Wave Equation

A function $f(x)$ is shown with a peak at f(0). Denote this by writing $f(0) = peak$. If we shift this function to the right by a distance d, then the new function $h(x)$ must be $h(x) = f(x-d)$. Here is how you can check this rule. Is the peak now at $x = d$? Does $h(d) = peak$? We check this below the figure.



$$f(0) = peak \quad \text{and} \quad h(x) = f(x-d)$$

$$h(d) = f(d-d) = f(0) = peak$$



It checks out. Do you remember doing this often in trigonometry? If you shift the cosine by $\pi/2$ to the right, you get the sine.

$$\sin x = \cos(x - \frac{\pi}{2})$$

The above relation also tells you that the sine of an angle in a right triangle equals the cosine of its complement.

Since $f(x-d)$ is our shifted function to the right by a distance d, we can let $d = vt$ to obtain a traveling function to the right. Let's search for a differential equation for this function, i.e., we want a differential equation such that our traveling wave $f(x-vt)$ is the solution. Common practice is to use ψ for a wave. So we write

$\psi(x,t) = f(x-vt)$, defining $u = x - vt$. Note that $\dfrac{\partial u}{\partial x} = 1$ and $\dfrac{\partial u}{\partial t} = -v$.

Then we take derivatives in our quest for the "magic" differential wave equation,

$$\frac{\partial \psi(x,t)}{\partial x} = \frac{\partial f(x-vt)}{\partial x} = \frac{\partial f(u)}{\partial x} = \frac{df(u)}{du}\frac{\partial u}{\partial x} = \frac{df(u)}{du}\cdot 1 = \frac{df(u)}{du}$$

$$\frac{\partial \psi(x,t)}{\partial t} = \frac{\partial f(x-vt)}{\partial t} = \frac{\partial f(u)}{\partial t} = \frac{df(u)}{du}\frac{\partial u}{\partial t} = \frac{df(u)}{du}\cdot(-v) .$$

We can now put together the following differential equation from the above. We find

$$\frac{\partial \psi(x,t)}{\partial x} = -\frac{1}{v}\frac{\partial \psi(x,t)}{\partial t} \text{ and write } \frac{\partial \psi_R(x,t)}{\partial x} = -\frac{1}{v}\frac{\partial \psi_R(x,t)}{\partial t} ,$$

adding the subscript R for "Right" to emphasize that this wave is traveling down the x axis in the positive direction.

But for the wave traveling to the left, we must have the same equation with the velocity in the negative direction. This reverses the sign in front of v since u in that case would be u = x + vt with f(u) = f(x+vt).

$$\frac{\partial \psi_L(x,t)}{\partial x} = +\frac{1}{v}\frac{\partial \psi_L(x,t)}{\partial t} .$$

This is not acceptable because now we have two differential equations and there is nothing special about right or left. We want a differential equation where the sign does not matter. So we proceed to the second derivative.

We start with

$$\psi(x,t) = f(x - vt) \quad \text{and} \quad u = x - vt \,,$$

$$\frac{\partial \psi(x,t)}{\partial x} = \frac{df(u)}{du} \quad \text{and} \quad \frac{\partial \psi(x,t)}{\partial t} = -v\frac{df(u)}{du} \,,$$

and take the second derivatives with respect to x and t.

$$\frac{\partial^2 \psi(x,t)}{\partial x^2} = \frac{\partial}{\partial x}\frac{df(u)}{du} = \frac{d^2 f(u)}{du^2}\frac{\partial u}{\partial x} = \frac{d^2 f(u)}{du^2}$$

$$\frac{\partial^2 \psi(x,t)}{\partial t^2} = \frac{\partial}{\partial t}\left[-v\frac{df(u)}{du}\right] = -v\frac{d^2 f(u)}{du^2}\frac{\partial u}{\partial t} = v^2\frac{d^2 f(u)}{du^2} \,.$$

This leads to

$$\boxed{\frac{\partial^2 \psi(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi(x,t)}{\partial t^2}}$$

Note that when you square plus or minus v that you get positive v squared. This differential equation applies to waves moving to the left or to the right. This is the wave equation in one dimension. The general solution is a combination of a wave moving right and one moving left:

$$\psi(x,t) = Af(x - vt) + Bg(x + vt)$$

For the wave equation in three dimensions where $\psi = \psi(x,y,z,t)$, we have

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

With the del operator ∇ , we can write this very elegantly. First note that since

$$\nabla \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k},$$

we have

$$\nabla \cdot \nabla = \left[\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right] \cdot \left[\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right]$$

$$\nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

We make the shorthand definition

$$\nabla^2 \equiv \nabla \cdot \nabla$$

The symbol ∇^2 is also called the Laplacian operator.

So

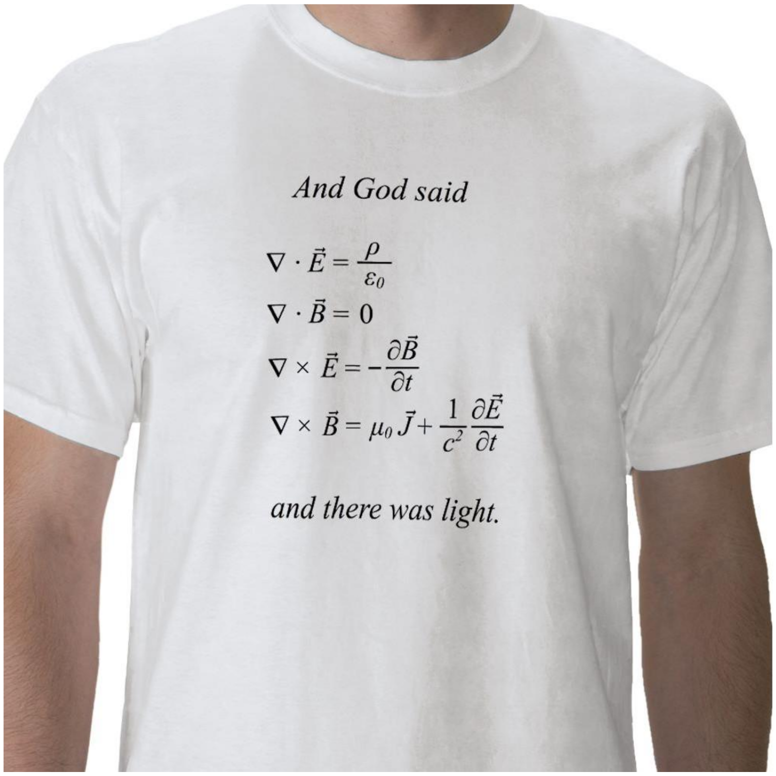
$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

can be neatly written as

$$\boxed{\nabla^2 \psi = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}}$$

You can remember where the v goes from dimensional analysis. Since distance equals velocity times time, your velocity has to go with the time t. Since we have the second derivative, think of distance as being squared and time as being squared. So you need the velocity squared.

F2. "Let There Be Light." Watch the video for a discussion of the variations below.



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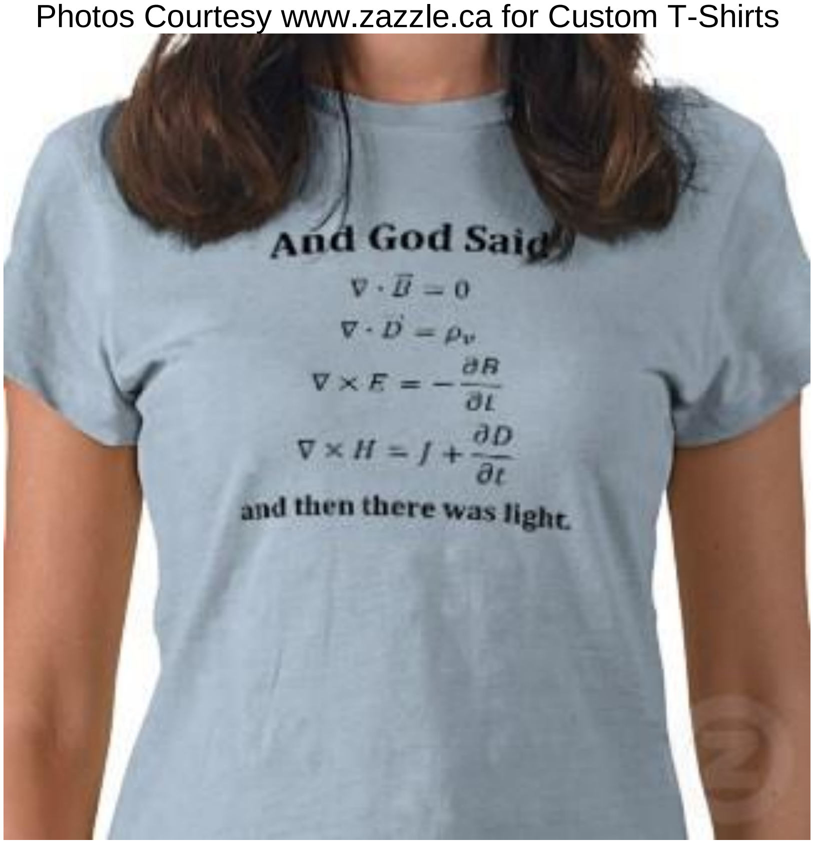


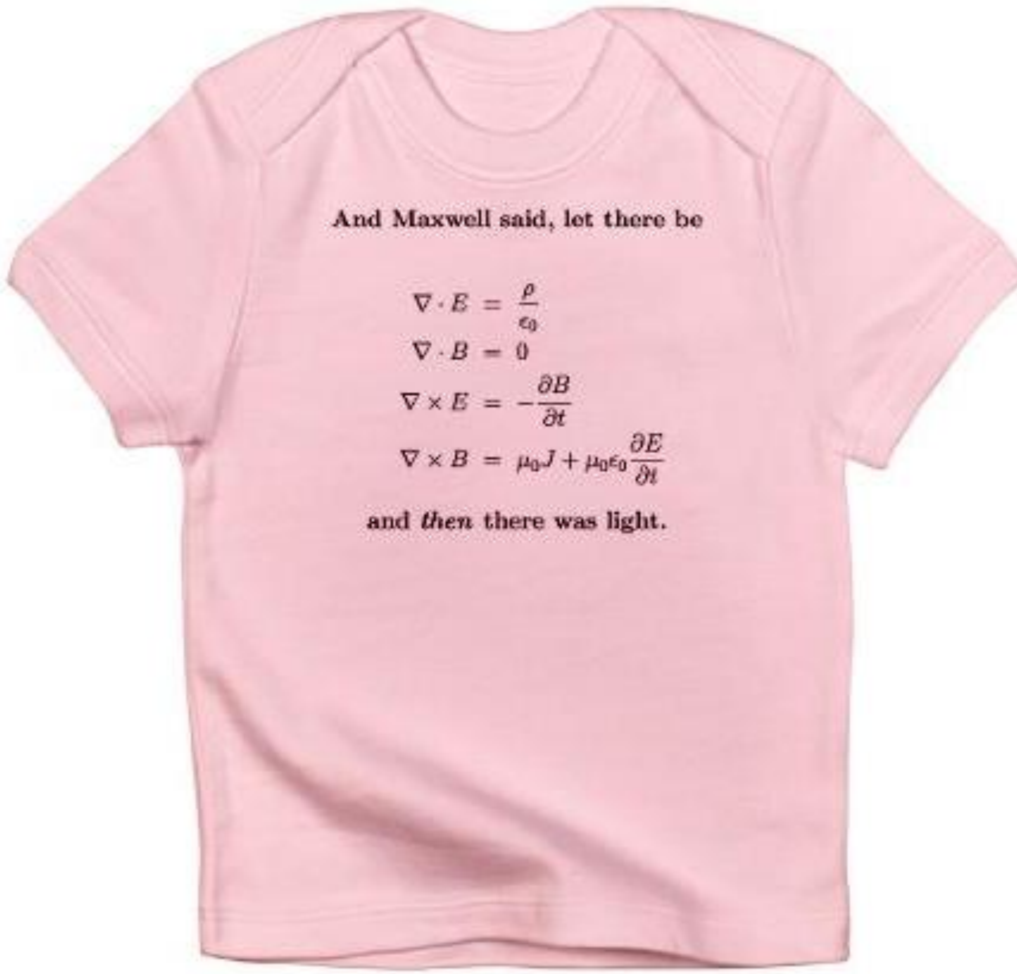


Photo Courtesy www.cafepress.com

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Free Space Equations

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

For the free space Maxwell equations we are far away from any charge sources and currents. Thus, we set

$\rho = 0$ and

$\vec{J} = 0$.

The free-space equations have beautiful symmetry and contain the secret about light. We play with these equations to see if a wave equation is supported. This is an example of theoretical physics at its best. We are in search of a discovery using theory only.

We are in search for a second order differential equation so we go for a second derivative with respect to time.

Take a derivative of the equation $\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ with respect to time.

$$\frac{\partial}{\partial t}(\nabla \times \vec{B}) = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla \times \frac{\partial \vec{B}}{\partial t} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

Now it's time to use the Maxwell equation with the $\frac{\partial \vec{B}}{\partial t}$, i.e., $(\nabla \times \vec{E}) = -\frac{\partial \vec{B}}{\partial t}$,

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E} \ .$$

Substituting this into our last equation gives us

$$\nabla \times (-\nabla \times \vec{E}) = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla \times (\nabla \times \vec{E}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

Let's focus on $\nabla\times(\nabla\times\vec{E})$. We do this by first calculating the curl of $\vec{E}$.

$$\nabla\times\vec{E}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ E_x & E_y & E_z\end{vmatrix}=\hat{i}(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z})-\hat{j}(\frac{\partial E_z}{\partial x}-\frac{\partial E_x}{\partial z})+\hat{k}(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y})$$

Then

$$\nabla\times(\nabla\times\vec{E})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \dfrac{\partial E_z}{\partial y}-\dfrac{\partial E_y}{\partial z} & \dfrac{\partial E_x}{\partial z}-\dfrac{\partial E_z}{\partial x} & \dfrac{\partial E_y}{\partial x}-\dfrac{\partial E_x}{\partial y}\end{vmatrix}.$$

Let's do the x-component first.

$$\left[\nabla\times(\nabla\times\vec{E})\right]_x=\frac{\partial}{\partial y}\left[\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right]-\frac{\partial}{\partial z}\left[\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right]$$

$$\left[\nabla\times(\nabla\times\vec{E})\right]_x=\frac{\partial^2 E_y}{\partial y\partial x}-\frac{\partial^2 E_x}{\partial y^2}-\frac{\partial^2 E_x}{\partial z^2}+\frac{\partial^2 E_z}{\partial z\partial x}$$

Flip the order of the derivatives for the first and last term to obtain

$$\left[\nabla\times(\nabla\times\vec{E})\right]_x=\frac{\partial^2 E_y}{\partial x\partial y}-\frac{\partial^2 E_x}{\partial y^2}-\frac{\partial^2 E_x}{\partial z^2}+\frac{\partial^2 E_z}{\partial x\partial z}.$$

$$\left[\nabla\times(\nabla\times\vec{E})\right]_x = \frac{\partial^2 E_y}{\partial x\partial y} + \frac{\partial^2 E_z}{\partial x\partial z} - \frac{\partial^2 E_x}{\partial y^2} - \frac{\partial^2 E_x}{\partial z^2}$$

$$\left[\nabla\times(\nabla\times\vec{E})\right]_x = \frac{\partial}{\partial x}\left[\frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right] - \frac{\partial^2 E_x}{\partial y^2} - \frac{\partial^2 E_x}{\partial z^2}$$

We now add to the right side zero in the form of $\frac{\partial^2 E_x}{\partial x^2} - \frac{\partial^2 E_x}{\partial x^2}$:

$$\left[\nabla\times(\nabla\times\vec{E})\right]_x = \frac{\partial}{\partial x}\left[\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right] - \frac{\partial^2 E_x}{\partial x^2} - \frac{\partial^2 E_x}{\partial y^2} - \frac{\partial^2 E_x}{\partial z^2} .$$

$$\nabla\times(\nabla\times\vec{E})_x = \frac{\partial}{\partial x}\left[\nabla\cdot\vec{E}\right] - \nabla^2 E_x$$

Note that we have discovered the following powerful identity:

$$\nabla\times(\nabla\times\vec{E}) = \nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E}$$

But $\nabla\cdot\vec{E} = 0$ in free space. Therefore:

$$\nabla\times(\nabla\times\vec{E})_x = -\nabla^2 E_x$$

There is nothing special about the x-direction. So the complete vector equation is

$\nabla\times(\nabla\times\vec{E}) = -\nabla^2\vec{E}$, consistent also from our above identity.

Putting it all to together, our equation

$\nabla\times(\nabla\times\vec{E}) = -\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$ becomes $\nabla^2\vec{E} = \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$.

Voilà! Compare this equation $\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$ to the wave equation

$$\frac{\partial^2 \psi(x,t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi(x,t)}{\partial t^2}$$

It is the wave equation for the electric field with $\frac{1}{v^2} = \mu_0 \varepsilon_0$.

Guess what Maxwell found for the speed $v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ when he put in the numerical values for μ_0 and ε_0? He found a value close to the then known value of the speed of light. This was in 1861. He concluded that light was an electromagnetic phenomenon. We will summarize our results below replacing the speed with the speed of light symbol.

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}, \text{ where } c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}.$$

Therefore,

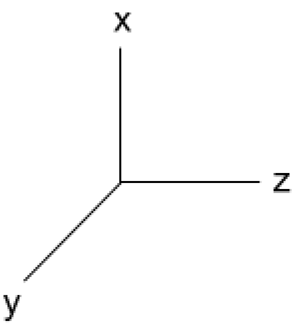
$$\boxed{\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}}$$

One can derive the same equation, but with the magnetic field replacing the electric field. It is faster now because we can use the powerful identity we derived, thus taking a shortcut. We would get. How, can we write this result down by clever comparison with the electric field case?

$$\boxed{\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}}.$$

Once again we find $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$.

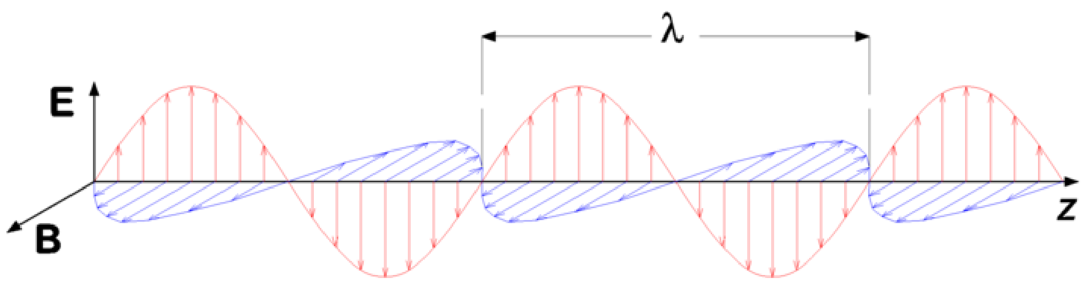
F3. Electromagnetic Waves



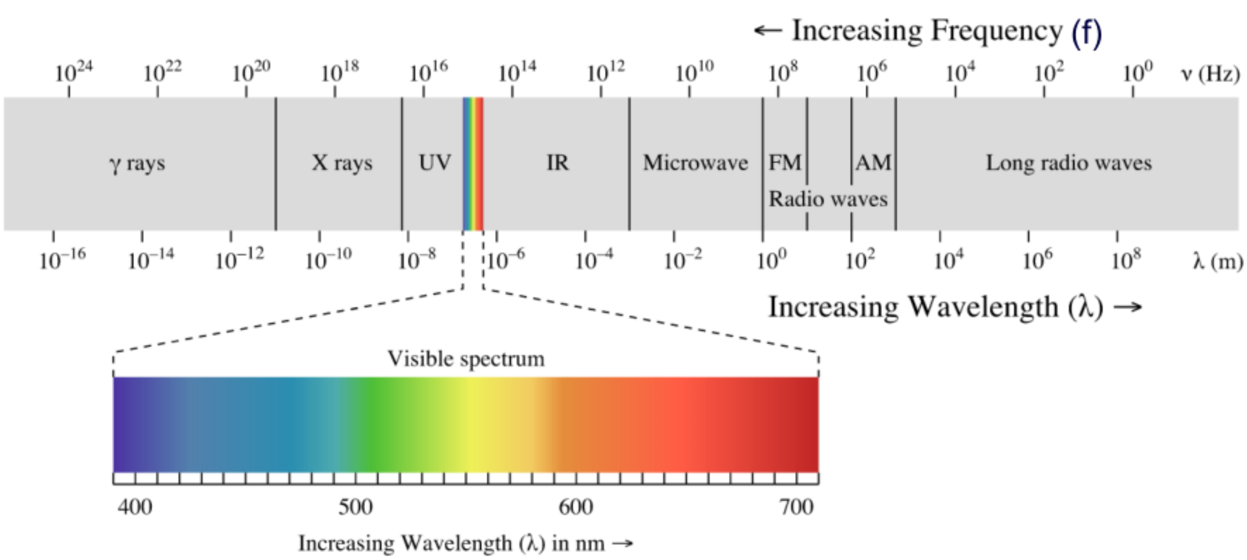
The axes at the left are defined with the usual association of the unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ with x, y, and z respectively. Note also that we have a right-handed system with

$$\hat{i} \times \hat{j} = \hat{k} \, .$$

For $\vec{E} = E_0 \sin\left[k(z - ct)\right]\hat{i}$, one can show that $\vec{B}$ is along the y axis with $\vec{B} = B_0 \sin\left[k(z - ct)\right]\hat{j}$, i.e., in phase with $\vec{E}$ and that $E_o / B_o = c$. The solution begins on the next page.

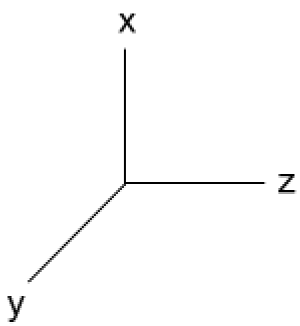


Courtesy P.wormer, Wikimedia



Adapted from Philip Ronan, Wikimedia

Start with a sinusoidal electric field vector traveling along the z axis. We will assume that the E field has a transverse and longitudinal, i.e., parallel, component. We orient the x axis so that the x axis aligns with the transverse component, i.e., the component perpendicular to the direction of propagation. Then we can write in general



$$\vec{E} = (E_x \hat{i} + E_z \hat{k}) \sin[k(z - ct)],$$

$$\vec{B} = (B_x \hat{i} + B_y \hat{j} + B_z \hat{k}) \sin[k(z - ct) + \delta],$$

where δ allows the magnetic field oscillations to be out of phase with respect to the oscillating electric fields. We will show that $\delta = 0$.

a) Using $\nabla \cdot \vec{E} = 0$, gives

$$\nabla \cdot \vec{E} = \left[\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right] \cdot (E_x \hat{i} + E_z \hat{k}) \sin[k(z - ct)] = 0$$

$$\nabla \cdot \vec{E} = \frac{\partial}{\partial x}\{E_x \sin[k(z - ct)]\}\hat{i} \cdot \hat{i} + \frac{\partial}{\partial z}\{E_z \sin[k(z - ct)]\}\hat{k} \cdot \hat{k} = 0$$

$$\nabla \cdot \vec{E} = (0)(\hat{i} \cdot \hat{i}) + kE_z \cos[k(z - ct)]\hat{k} \cdot \hat{k} = 0$$

$$kE_z \cos[k(z - ct)] = 0$$

$E_z = 0$ in order to make this equation true always.

b) Using $\nabla \cdot \vec{B} = 0$ will similarly give $B_z = 0$.

c) We will calculate $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ by first working with $\nabla \times \vec{E}$. Note that

$$\vec{E} = E_x \sin\left[k(z-ct)\right]\hat{i}$$

as we have shown that $E_z = 0$. Then,

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x \sin\left[k(z-ct)\right] & 0 & 0 \end{vmatrix}$$

$$\nabla \times \vec{E} = kE_x \cos\left[k(z-ct)\right]\hat{j}$$

For the magnetic field we have so far $\vec{B} = (B_x \hat{i} + B_y \hat{j})\sin\left[k(z-ct)+\delta\right]$

$$-\frac{\partial \vec{B}}{\partial t} = -(B_x \hat{i} + B_y \hat{j})\cos\left[k(z-ct)+\delta\right](-kc)$$

$$-\frac{\partial \vec{B}}{\partial t} = kc(B_x \hat{i} + B_y \hat{j})\cos\left[k(z-ct)+\delta\right]$$

We comparing the two equations, which must be equal.

$$\nabla \times \vec{E} = kE_x \cos\left[k(z-ct)\right]\hat{j}$$

$$-\frac{\partial \vec{B}}{\partial t} = kc(B_x \hat{i} + B_y \hat{j})\cos\left[k(z-ct)+\delta\right]$$

Then, $B_x = 0$, $B_y = \frac{E_x}{c}$, and $\delta = 0$.