# Theoretical Physics
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# Chapter F Solutions. Differential Form for the Maxwell Equations

**HW-F1. The Wave Equation for the Magnetic Field.** Use the differential form of the free-space Maxwell equations to derive the wave equation

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$ using the shortcut identity

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} \, .$$

## Solution

Free Space Equations

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Start with $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ and take the curl of both sides.

$$\nabla \times (\nabla \times \vec{E}) = -\nabla \times \frac{\partial \vec{B}}{\partial t}$$

Since space and time derivatives can be interchanged,

$$\nabla \times (\nabla \times \vec{E}) = -\frac{\partial \nabla \times \vec{B}}{\partial t} \, .$$

Now use the identity $\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$ .

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial \nabla \times \vec{B}}{\partial t}$$

One of the free-space Maxwell equations is $\nabla \cdot \vec{E} = 0$, leading to

$$-\nabla^2 \vec{E} = -\frac{\partial \nabla \times \vec{B}}{\partial t}.$$

Another free-space Maxwell equation is , $\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ , leading to

$$-\nabla^2 \vec{E} = -\frac{\partial}{\partial t}(\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}).$$

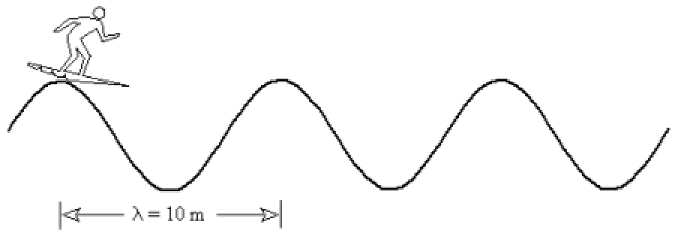
$$\nabla^2 \vec{E} = \frac{\partial}{\partial t}(\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t})$$

$$\boxed{\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}}$$

This equation is the wave equation with speed $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$.

**HW-F2. The Wave Relation.** An ideal surfer rides on the crest of a wave.

a. If the waves are 10 meters apart and 5 crests go by second, what is the speed of the wave?




Let the distance from one crest to the next, your 10 m, be called the wavelength $\lambda$. Then let the number of crests per second be called the frequency f. Give the relation from your analysis that relates $\lambda$, f, and v (the speed of the wave). The common unit for frequency is hertz, where hertz = 1/s. Explain why this must be the case for your units to come out correctly.

The speed of the wave is 50 meters per second. The general relation is

$$v = \lambda f \; .$$

In order to obtain velocity of units of meters per second from wavelength you have to divide the wavelength by time.Therefore, frequency must have units of 1 over time.

b. Now consider sound waves where v = 340 m/s at room temperature and pressure. What is the wavelength in centimeters (cm) for a 440-Hz sound wave, the note they use to tune an orchestra? See the discussion on significant figures below.

$$\lambda = \frac{v}{f} = \frac{340\,\frac{m}{s}}{440\,\frac{1}{s}} = 0.77\;m = 77\;cm \; .$$

c. Now consider light waves where c = 300,000 km/s, where km is kilometer, i.e., 1000 meters. What is the wavelength in cm for a 30-GHz microwave? Note that the metric prefix G stands for Giga, which is equal to one billion.

$$\lambda = \frac{c}{f} = \frac{300,000,000\ \dfrac{\mathrm{m}}{\mathrm{s}}}{30,000,000,000\ \dfrac{1}{\mathrm{s}}} = \frac{300\ \mathrm{m}}{30,000} = \frac{1}{100}\ \mathrm{m} = 1\ \mathrm{cm}$$

d. What is the frequency in hertz for a 645-nm red diode laser? Note that the metric prefix n stands for nano, which is equal to one billionth. Give your answer to three significant figures. Remember, never give more significant figures than the most uncertain measurement. Trailing zeros typically do not count as significant figures, but the speed of light given above is accurate to four significant figures.

$$f = \frac{c}{\lambda} = \frac{3\times 10^{8}\ \dfrac{\mathrm{m}}{\mathrm{s}}}{645\times 10^{-9}\ \mathrm{m}} = \frac{3\times 10^{8}}{6.45\times 10^{-7}}\ \mathrm{Hz} = 0.465\times 10^{15}\ \mathrm{Hz} = 4.65\times 10^{14}\ \mathrm{Hz}$$

Using the exact $c = 299,792,458\ \mathrm{m/s}$ ,

$$f = \frac{c}{\lambda} = \frac{2.99792458\times 10^{8}}{6.45\times 10^{-7}}\ \mathrm{Hz} = 0.465\times 10^{15}\ \mathrm{Hz} = 4.65\times 10^{14}\ \mathrm{Hz}\ .$$