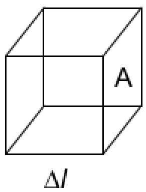
Theoretical Physics
Prof. Ruiz, UNC Asheville
Chapter G Homework. Ideal Gas Law and Thermodynamics

HW-G1. Deriving PV = nRT for an Intro Course. Consider a gas in a cube where each particle travels at the same speed v. Also, at any given time 1/3 of the particles are moving in each of the 3 dimensions. Then make your model more general by considering N_1 particles with velocity v_1, N_2 particles with velocity v_2, , etc.



Let N represent the total number of particles in the box.

Solution. Pick one of the six walls to analyze. We will consider the left wall. Freeze the moment. Then $\frac{N}{6}$ of the particles are heading to the left wall. Let time Δt be the time for a particle at the extreme right to travel to the left the full Δl to reach the left wall. In this time, all the particles will have collided with the left wall.

The change in the momentum for each particle is

$$\Delta p = p_{final} - p_{initial} = mv - (-mv) = 2mv .$$

Note that all particles travel with the same speed. Therefore $v = \frac{\Delta l}{\Delta t}$. The total force is given by

$$F = \frac{N}{6}\frac{\Delta p}{\Delta t} = \frac{N}{6}\frac{2mv}{\Delta t} = \frac{N}{3}\frac{mv}{\Delta t} .$$

Pressure is force per unit area.

$$P = \frac{F}{A} = \frac{N}{3}\frac{mv}{A\Delta t}$$

Multiply the right side by $1 = \frac{\Delta l}{\Delta l}$ to obtain $P = \frac{N}{3}\frac{mv}{A\Delta l}\frac{\Delta l}{\Delta t} = \frac{N}{3}\frac{mv}{V}v = \frac{N}{3}\frac{mv^2}{V}$, which gives the following.

$$\boxed{PV = \frac{N}{3}mv^2}$$

Since $\frac{1}{2}m\overline{v^2} = \frac{3}{2}kT = \frac{1}{2}mv^2$ (all speeds are the same),

$$\boxed{PV = NkT} \quad \text{or} \quad \boxed{PV = nRT}.$$

Note that in class we found $PV = \frac{N}{3} m\overline{v^2}$. But since all particles have the same speed, the average speed $\bar{v} = v$ and average velocity squared is $\overline{v^2} = v^2$.

Generalizing more, following the same steps for each velocity, and adding the results:

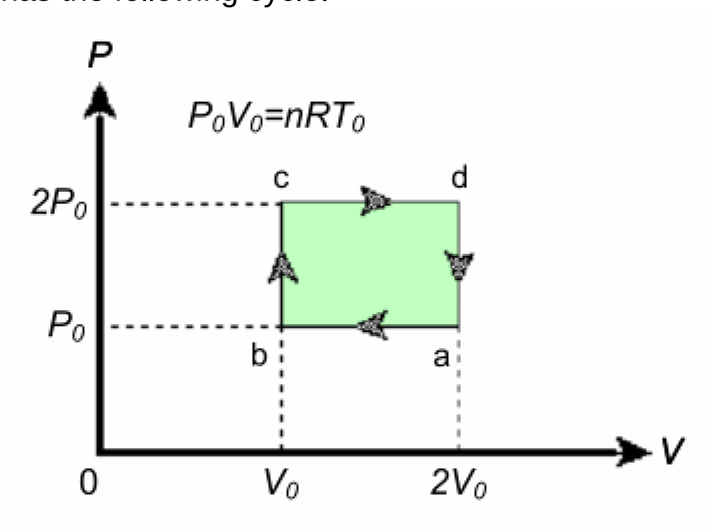
$$PV = \frac{N}{3} mv^2 \text{ generalizes to } PV = \frac{m}{3}\left[N_1 v_1^2 + N_2 v_2^2 + N_3 v_3^2 + \ldots\right].$$

$$PV = \frac{N}{3} m\left[\frac{N_1}{N} v_1^2 + \frac{N_2}{N} v_2^2 + \frac{N_3}{N} v_3^2 + \ldots\right]$$

$$\boxed{PV = \frac{N}{3} m\overline{v^2}}$$

Since $\frac{1}{2} m\overline{v^2} = \frac{3}{2} kT$, we arrive at $\boxed{PV = NkT}$ or $\boxed{PV = nRT}$.

HW-G2. A Simple Engine. An ideal engine with an ideal gas, i.e., where $PV = nRT$, has the following cycle.



Fill in the table; find the efficiency.

Calculate $\Delta W = P\Delta V$ first (far right column). It is the area under the graph for each phase. Note the no change in volume for b to c and d to a. Then do energy (1st column).

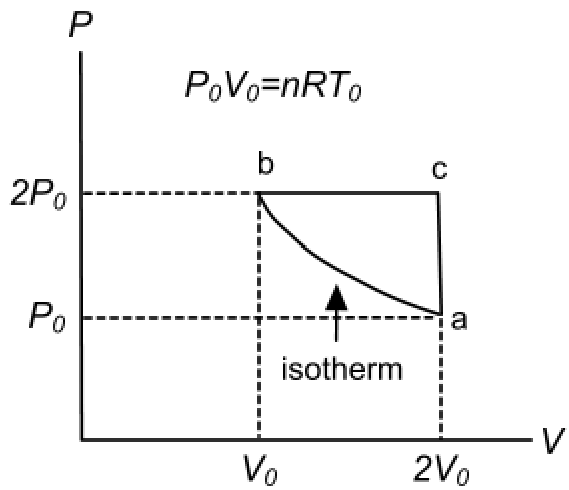
Temperatures: $T_b = T_0$ (given) and $T_a = T_c = 2T_0$, $T_d = 4T_0$ from $PV = nRT$.

	$\Delta U = \frac{3}{2}nR\Delta T$	$\Delta Q = \Delta U + P\Delta V$	$\Delta W = P\Delta V$
a to b	$-\frac{3}{2}nRT_0$	$-\frac{5}{2}nRT_0$	$-P_0V_0 = -nRT_0$
b to c	$\frac{3}{2}nRT_0$	$\frac{3}{2}nRT_0$	0
c to d	$\frac{3}{2}nR2T_0 = 3nRT_0$	$5nRT_0$	$2P_0V_0 = nR2T_0$
d to a	$-\frac{3}{2}nR2T_0 = -3nRT_0$	$-3nRT_0$	0

Efficiency: $\eta = \frac{W}{Q_{in}}$, where W is the net work performed and Q_{in} is the input heat. The net work done is the area inside the box: $W = P_0V_0 = nRT_0$, which you can also arrive at by summing all the ΔW values in the last column in our table. The Q_{in} is equal to the sum of the two positive ΔQ values in the middle column. Therefore,

$$W = P_0V_0 = nRT_0 \text{ and } \Delta Q = \frac{3}{2}nRT_0 + 5nRT_0 = \frac{13}{2}nRT_0.$$

The efficiency is $\eta = \frac{W}{Q_{in}} = \frac{nRT_0}{(13/2)nRT_0} = \frac{2}{13}$.



HW-G3. Simple Engine With Isotherm.
An engine with an ideal a-b-c cycle.

First show that the pressure and volume at the endpoints "a" and "b" satisfy the equation that describes an isothermal process for an ideal gas.

Calculate the work for each of the three phases of the cycle Fill these in the usual table in terms of n, R, and T_0.

Finally, calculate the efficiency of the engine. Determine temperatures first.

a: $PV = nRT$ gives $P_0 2V_0 = nRT_a = 2nRT_0$ since $P_0V_0 = nRT_0$ is given. $T_a = 2T_0$

b. $2P_0V_0 = nRT_b = 2nRT_0$. Therefore, $T_b = T_a$, i.e., **the isotherm endpoints.**

c: $PV = nRT$ gives $2P_0 2V_0 = nRT_c = 4nRT_0$. Therefore, $T_c = 4T_0$.

$$\Delta W_{a\to b} = \int_a^b PdV = \int_a^b \frac{nRT_a}{V}dV = nRT_a\int_a^b \frac{dV}{V} = nRT_a \ln V\Big|_{2V_0}^{V_0} = nR2T_0 \ln\frac{1}{2} = -2nRT_0 \ln 2$$

$\Delta W_{b\to c} = P\Delta V = 2P_0(2V_0 - V_0) = 2P_0V_0 = 2nRT_0$ and $\Delta W_{c\to a} = 0$ (isometric)

You can get the middle column from $\Delta U = \Delta Q - P\Delta V$, i.e., $\Delta Q = \Delta U + P\Delta V$.

	ΔU	ΔQ	ΔW
a to b	0 isotherm	$-2nRT_0 \ln 2$	$-2nRT_0 \ln 2$
b to c	$\frac{3}{2}nR\Delta T_{b\to c} = 3nRT_0$	$5nRT_0$	$2nRT_0$
c to a	$\frac{3}{2}nR\Delta T_{c\to a} = -3nRT_0$	$-3nRT_0$	0

Efficiency $\eta = \dfrac{W}{Q_{in}} = \dfrac{-2nRT_0 \ln 2 + 2nRT_0}{5nRT_0} = \dfrac{2 - 2\ln 2}{5} = \dfrac{0.614}{5} = 0.12$.