Theoretical Physics
Prof. Ruiz, UNC Asheville, doctorphys on YouTube
Chapter M Notes. The Method of Frobenius

M1. The Method of Frobenius.

Ferdinand Georg Frobenius (1849-1917)
Courtesy School of Mathematics & Statistics
University of St. Andrews, Scotland

This approach to solving differential equations consists of four general steps.

Step 1. Series Plug In. First, you assume a solution in the form of a simple power series.

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

If this does not work, one can try k + r. After that, there are other tricks one learns about in differential equations.

Plug into your equation y(x), y'(x), and y''(x) .

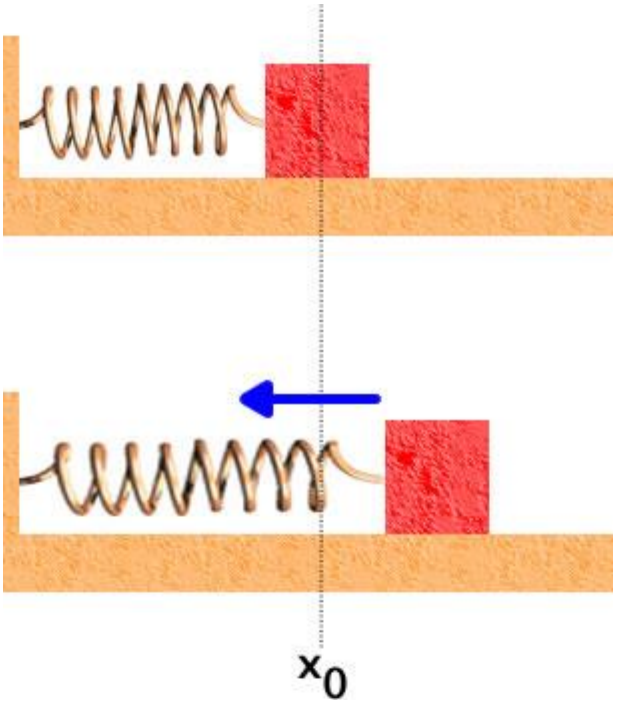
$$y'(x) = \sum_{k=0}^{\infty} k a_k x^{k-1} \quad \text{and} \quad y''(x) = \sum_{k=0}^{\infty} k(k-1) a_k x^{k-2}$$

You could let k start at 1 in the y' sum since k = 0 gives zero anyway. Similarly, you could start k at 2 in the second sum since k = 0 and k = 1 both give zero anyway.

Step 2. Fix the Exponents. Shift k in some terms if needed and relabel so that all the terms go from $k = 0$ to $k = \infty$ in "sync" with each other.
.
Step 3. The Arbitrary Argument. Get everything on one side of the equation and factor out x^k. Now use the arbitrary argument to set everything else to zero term by term in your sum. The arbitrary argument states that if your infinite sum happens to be zero for a particular value of x, it surely will not be for an arbitrary value of x.

Step 4. The Recursion Relation. Find a recurrence relation for the coefficients a_k.

M2. Equations Encountered in Physics. Consider Newton's Second Law.



A Mass is Attached to Spring.
Courtesy David M. Harrison
Department. of Physics
University of Toronto

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt} = ma$$

Hooke's Law gives the force equal to

$$F_k = -kx\,.$$

For a retarding frictional force proportional to the velocity:

$$F_b = -bv$$

Putting this all together, $F = F_k + F_b$, and we get.

$$-kx - bv = ma$$

$$ma + bv + kx = 0$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

Equations in classical mechanics have a second derivative, first derivation, and zeroth derivative. Using y = y(x), a general form mathematicians study is relevant to physics.

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0$$

This also covers the time-independent Schrödinger equation, which is a second-order differential equation. In our course on the various areas of physics we have never encountered a third derivative. The Dirac equation is first order. So we are all set. We will consider a sample differential equation that has this form in our next section.

M3. The Legendre Differential Equation.

Adrien-Marie Legendre (1752-1833)
From Wikipedia: 1820 watercolor caricature of Adrien-Marie Legendre by French artist Julien-Leopold Boilly. It's the only existing portrait.

The Legendre differential equation is

$$(1-x^2)y''-2xy'+l(l+1)y=0\,,$$

where $l=0,1,2,3,\ldots$

When we solve the Schrödinger equation for the hydrogen atom in spherical coordinates in quantum mechanics, we come up with the equation in the following form where p is some constant.

$$(1-x^2)y''-2xy'+py=0$$

Let's use this latter form and begin with the method of Frobenius. During our solution, you will see how the $l(l+1)$ comes about.

Step 1. "Series Plug In." We assume a solution in the form of a power series: $y(x)=\sum_{k=0}^{\infty}a_k x^k$. The we calculate derivatives. We are going to keep k = 0 for each since k = 0 gives zero anyway in the y' equation and k = 0 or k = 1 both give zero anyway in the y" series.

$$y'=\sum_{k=0}^{\infty}k a_k x^{k-1} \quad \text{and} \quad y''(x)=\sum_{k=0}^{\infty}k(k-1)a_k x^{k-2}$$

We can stop taking derivatives since in our case we have a second-order differential equation that requires up to the second derivative. We plug in the power series for y, y' and y'' Then $(1-x^2)y''-2xy'+py=0$ becomes

$$(1-x^2)\sum_{k=0}^{\infty}k(k-1)a_k x^{k-2}-2x\sum_{k=0}^{\infty}k a_k x^{k-1}+p\sum_{k=0}^{\infty}a_k x^k=0\,.$$

We want to collect the "x" variables in one place in each of the summations so everything is transparent for us.

$$\sum_{k=0}^{\infty} k(k-1)a_k x^{k-2} - \sum_{k=0}^{\infty} k(k-1)a_k x^k - 2\sum_{k=0}^{\infty} ka_k x^k + p\sum_{k=0}^{\infty} a_k x^k = 0$$

Step 2. "Fix the Exponents." We need to adjust the exponents and the k-starting point so that we have x^k in each term. The only summation needing adjustment is the first one. Let $m = k-2$, i.e., $k = m+2$ so we can write

$$\sum_{k=0}^{\infty} k(k-1)a_k x^{k-2} = \sum_{m=-2}^{\infty} (m+2)(m+1)a_{m+2} x^m .$$

Note that we can start m at $m = 0$ since you get zero for $m = -2$ and $m = -1$ anyway.

$$\sum_{m=-2}^{\infty} (m+2)(m+1)a_{m+2} x^m = \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m$$

The variable "m" is our "summation variable" since it is summed over, i.e., forced to take on each integer value of the infinite sum. So any index can be used. Therefore, without any fuss, we simply replace it with "k".

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m = \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k$$

Putting this back into our differential equation, we have

$$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k - \sum_{k=0}^{\infty} k(k-1)a_k x^k - 2\sum_{k=0}^{\infty} ka_k x^k + p\sum_{k=0}^{\infty} a_k x^k = 0$$

Step 3. "The Arbitrary Trick." We pull out the common x^k factor, arriving at

$$\sum_{k=0}^{\infty} \left[(k+2)(k+1)a_{k+2} - k(k-1)a_k - 2ka_k + pa_k \right] x^k = 0 .$$

Since the x^k factor is arbitrary, i.e., we can choose x to be what we would like, the expression in the brackets must vanish for each k.

$$(k+2)(k+1)a_{k+2} - k(k-1)a_k - 2ka_k + pa_k = 0$$

This means

$$(k+2)(k+1)a_{k+2} = \left[k(k-1)+2k-p\right]a_k$$

Note that

$$k(k-1)+2k = k^2 - k + 2k = k^2 + k = k(k+1)$$

Therefore,

$$(k+2)(k+1)a_{k+2} = \left[k(k+1)-p\right]a_k$$

Step 4. "The Recursion Relation." Our final step is a recursion (or recurrence) relation where a_{k+2} is given in terms of a_k.

$$a_{k+2} = \left[\frac{k(k+1)-p}{(k+1)(k+2)}\right]a_k$$

And now we see that our series will run wild forever unless p is such that for some k eventually,

$$k_{\max}(k_{\max}+1) - p = 0 .$$

This will happen if

$$p = l(l+1) \text{, where } l = 0,1,2,...$$

For each value of l we have a polynomial. These polynomials corresponding to the l values are known as Legendre polynomials. You now see why the L:egendre differential equation is given with $p = l(l+1)$.

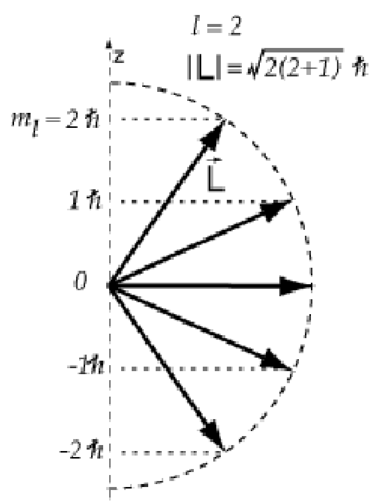
We come to the crossroads where mathematics (represented by Legendre, left) meets physics (Schrödinger, right).

Math: Let $p = l(l+1)$ with l = 0, 1, 2, 3 ... so that we get polynomials to study.

Physics: Boundary conditions force wave functions to be finite, i.e., a finite series.

In either case, the series must be terminated. Legendre had already taken care of this.

In quantum mechanics, the differential equation we are investigating is rich in orbital physics for a general spherically-symmetric potential of the form V = V(r). The requirement that we must have a finite probability distribution can be thought of as natural quantization coming from the Schrödinger equation. Often, this requirement is referred to as a boundary condition.



Angular Momentum
Courtesy Ron Nave, Hyper Physics

This particular requirement in the context of the physics here leads to the quantization of the total angular momentum. But we are not going to derive this result. We just mention this for you to be on the lookout for it if you take quantum mechanics.

$$L^2 = \hbar^2 l(l+1)$$

Note that the quantization along the z-axis gives Bohr's version postulated in 1913. You can only choose one axis with the total angular momentum in order to have a common set of eigenstates (i.e., for L_z and **L**).

M4. The Legendre Polynomials.

These correspond to your solutions $y = y(x)$ where $l = 0, 1, 2, ...$ and they are designated by $P_l(x)$. Note that when l is even, the even series terminates at some point but the odd series does not. So we will pick a_1 to be zero for the even cases of l. Similarly, we choose a_0 to be zero for the odd cases of l. Finally, by convention, we choose the nonzero a_0 or a_1 such that $P_l(1) = 1$. Keep the recurrence formula in front of you for all these calculations.

$$a_{k+2} = \left[\frac{k(k+1) - l(l+1)}{(k+1)(k+2)} \right] a_k$$

0. The Zeroth Legendre Polynomial ($l = 0$).

For $l = 0$ we set $a_1 = 0$ and note that $a_0 \neq 0$. Our recursions relation is

$$a_{k+2} = \left[\frac{k(k+1) - 0(0+1)}{(k+1)(k+2)} \right] a_k = \left[\frac{k(k+1)}{(k+1)(k+2)} \right] a_k$$

So we are off to find a_2. We do this by setting k = 0.

$$a_{0+2} = \left[\frac{0(0+1) - 0(0+1)}{(0+1)(0+2)} \right] a_0$$, which leads to $a_2 = 0$.

The zeroth Legendre polynomial is

$$P_0(x) = a_0 .$$

With the convention that for all the Legendre polynomials $P_l(1) = 1$, we have $a_0 = 1$ and therefore,

$$P_0(x) = 1$$

1. The First Legendre Polynomial ($l=1$).

$$a_{k+2} = \left[\frac{k(k+1) - l(l+1)}{(k+1)(k+2)} \right] a_k$$

For $l=1$ we set $a_0 = 0$ and note that $a_1 \neq 0$. Our recurrence relation is

$$a_{k+2} = \left[\frac{k(k+1) - 1(1+1)}{(k+1)(k+2)} \right] a_k = \left[\frac{k(k+1) - 2}{(k+1)(k+2)} \right] a_k$$

So we are off to find a_3. We do this by setting k = 1.

$$a_{1+2} = \left[\frac{1(1+1) - 2}{(1+1)(1+2)} \right] a_1$$, which leads to $a_3 = 0$.

The first Legendre polynomial is

$$P_1(x) = a_1 x \,.$$

With the convention that for all the Legendre polynomials $P_0(1) = 1$, we have $a_1 = 1$ and therefore,

$$P_1(x) = x$$

We will proceed to the next polynomial on the following page.

2. The Second Legendre Polynomial ($l = 2$).

For $l = 2$ we set $a_1 = 0$ and note that $a_0 \neq 0$. Our recurrence relation is

$$a_{k+2} = \left[\frac{k(k+1) - 2(2+1)}{(k+1)(k+2)}\right] a_k = \left[\frac{k(k+1) - 6}{(k+1)(k+2)}\right] a_k$$

So we are off to find a_2. We do this by setting k = 0.

$$a_{0+2} = \left[\frac{0(0+1) - 6}{(0+1)(0+2)}\right] a_0 = \frac{0-6}{(1)(2)} a_0 = -3a_0$$

Then for a_4 set k = 2.

$$a_4 = a_{2+2} == \left[\frac{2(2+1) - 6}{(2+1)(2+2)}\right] a_2 = 0$$

We are finished. The second Legendre polynomial is

$$P_2(x) = a_0 + a_2 x^2 = a_0 - 3a_0 x^2 = a_0(1 - 3x^2).$$

With the convention that for all the Legendre polynomials $P_l(1) = 1$, we must have

$$P_2(1) = a_0(1-3) = -2a_0 = 1 \quad \text{and} \quad a_0 = -\frac{1}{2}.$$

Therefore,

$$P_2(x) = -\frac{1}{2}(1 - 3x^2).$$

Writing the polynomial with the highest power first, we have

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

3. The Third Legendre Polynomial ($l = 3$).

For $l = 3$ we set $a_0 = 0$ and note that $a_1 \neq 0$. Our recurrence relation is

$$a_{k+2} = \left[\frac{k(k+1) - 3(3+1)}{(k+1)(k+2)} \right] a_k = \left[\frac{k(k+1) - 12}{(k+1)(k+2)} \right] a_k$$

So we are off to find a_3. We do this by setting k = 1.

$$a_{1+2} = \left[\frac{1(1+1) - 12}{(1+1)(1+2)} \right] a_1 = \frac{2-12}{(2)(3)} a_1 = -\frac{10}{6} a_1 = -\frac{5}{3} a_1$$

$$a_3 = -\frac{5}{3} a_1$$

Then for a_5 set k = 3.

$$a_5 = a_{3+2} = \left[\frac{3(3+1) - 12}{(3+1)(3+2)} \right] a_3 = 0$$

We are finished with the recurrence relation. The third Legendre polynomial is shown below, but we need to still find the constant.

$$P_3(x) = a_1 x + a_3 x^3 = a_1 x - \frac{5}{3} a_1 x^3 = a_1 (x - \frac{5}{3} x^3)$$

With the convention that for all the Legendre polynomials $P_l(1) = 1$, we must have

$$P_3(1) = a_1 (1 - \frac{5}{3}) = -\frac{2}{3} a_1 = 1 \text{ and } a_1 = -\frac{3}{2} \text{ giving } P_3(x) = -\frac{3}{2} (x - \frac{5}{3} x^3)$$

$$P_3(x) = \frac{1}{2} (5x^3 - 3x)$$

PM1 (Practice Problem). Find the 4th and 5th Legendre polynomials.