Theoretical Physics
Prof. Ruiz, UNC Asheville, doctorphys on YouTube
Chapter N Notes. The Dirac Delta Function

N1. The Dirac Delta Function.

Paul Dirac (1902-1984)
Courtesy School of Mathematics and Statistics
University of St. Andrews, Scotland

Consider the following crazy function.

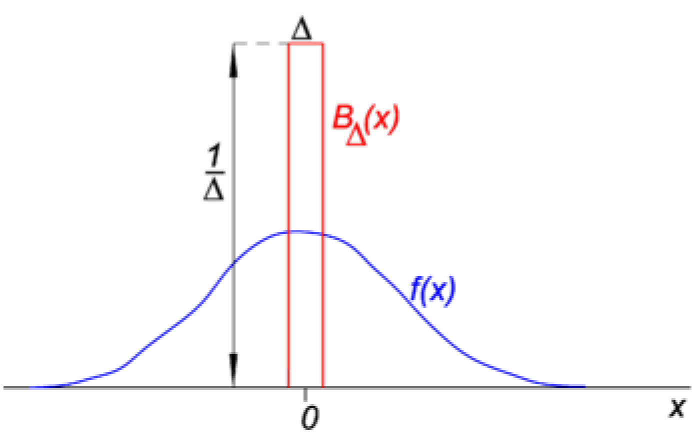
$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ +\infty, & x = 0 \end{cases}$$

with

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

No, no, no! You can't do this. How can you even calculate an area where there is no thickness? We will address this shortly. But first, the box "function."

The Box Function B(x). Courtesy Citizendium. The Box is centered over x = 0.



$B_\Delta(x) = 0$ outside the Δ region

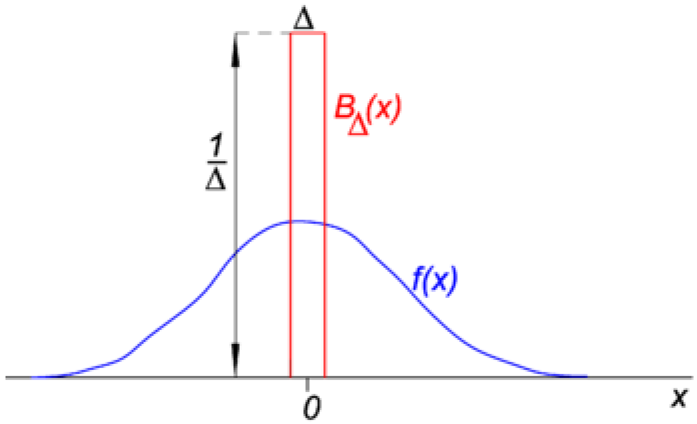
$B_\Delta(x) = \dfrac{1}{\Delta}$ in the Δ region

The area for this function is

$$\int_{-\infty}^{\infty} B_\Delta(x) dx = \frac{1}{\Delta}\Delta = 1$$

The family of box functions all have area 1. As Δ gets smaller and smaller, the box gets taller and thinner. Then,

$$\lim_{\Delta \to 0} B_\Delta(x) = \delta(x) .$$



The Sifting Property. We integrate the box function with an arbitrary function f(x). Then, by the mean value theorem for integration,

$$\int_{-\infty}^{\infty} B_\Delta(x) f(x) dx \overset{MVT}{=} \frac{1}{\Delta} f(c)\Delta,$$

where x = c is somewhere in the Δ region. The mean value theorem for integrals states that we can find some average height f(c) in the Δ region so that f(c) Δ gives the area of the product function for our strip. Therefore,

$$\int_{-\infty}^{\infty} B_\Delta(x) f(x) dx \overset{MVT}{=} f(c)$$

If we make the box thinner and thinner, we squeeze "c" to be zero.

$$\lim_{\Delta \to 0} \int_{-\infty}^{\infty} B_\Delta(x) f(x) dx \overset{MVT}{=} \lim_{\Delta \to 0} f(c) = f(0)$$

But for the super tall, super thin box, we have our delta function.

$$\lim_{\Delta \to 0} \int_{-\infty}^{\infty} B_\Delta(x) f(x) dx = \int_{-\infty}^{\infty} \delta(x) f(x) dx$$

So we write the shifting result as

$$\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0).$$

PN1 (Practice Problem). Show that

$$\int_{-\infty}^{\infty} \delta(x-a) f(x) dx = f(a).$$

Hint. Use a change of variable z = x – a and what you know so far about δ(x).

N2. The Dartboard and the Gaussian.

We are going to be using a sequence of Gaussians to clean things up mathematically. Remember from the beginning of our course the Gaussian integral we did with the polar coordinates trick?

$$I = \int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx = \sqrt{\frac{\pi}{\alpha}}$$

Then we used the derivative trick to evaluate a similar integral with x^2 included in the integrand.

$$I = \int_{-\infty}^{\infty} x^2 e^{-ax^2}\, dx = -\frac{d}{d\alpha}\int_{-\infty}^{\infty} e^{-ax^2}\, dx = -\frac{d}{d\alpha}\sqrt{\frac{\pi}{\alpha}} = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}$$

The Gaussian is of utmost importance in statistics. Anything you measure in large populations that vary from an average such as heights and weights tend towards a Gaussian.

To illustrate how the Gaussian comes up in statistics so readily we will present for you a nice example worked out by Dr. Dan Teague of our own *North Carolina School of Science and Mathematics (NCSSM)* in Durham, NC. That *NCSSM* is one impressive high school!

The Bandit Dart Board
Made by DMI (www.darts.com)
Courtesy amazon.com

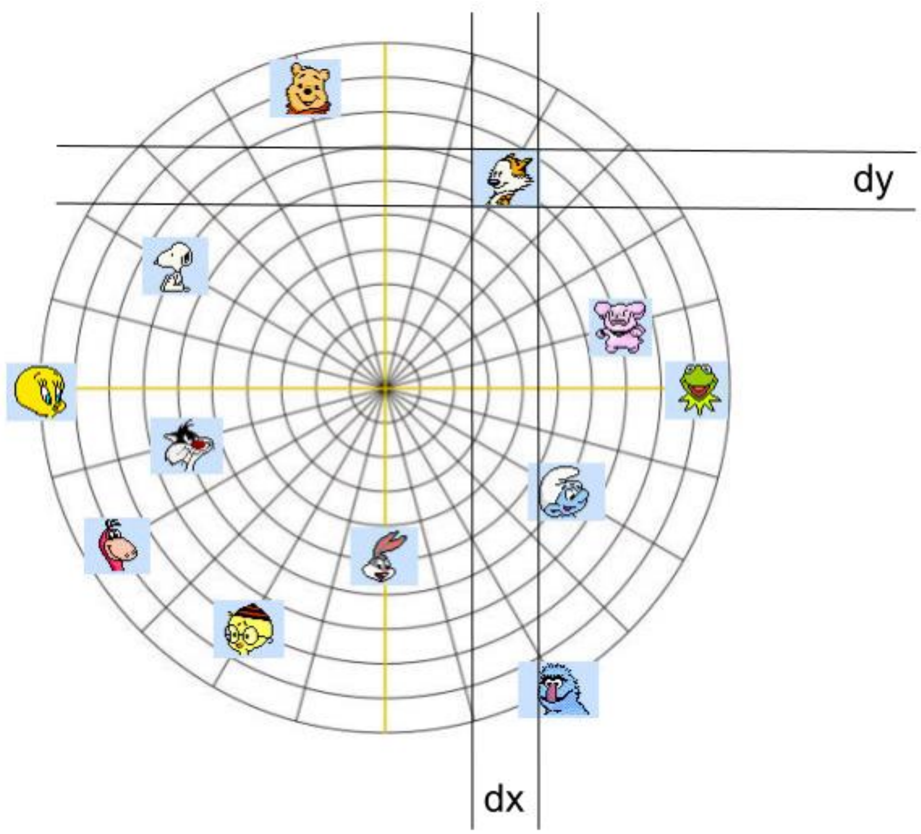
We will be using two statistics definitions. One is your average (or mean) μ and the other is a measure of the spread of values from the average called σ^2, the variance.

$$\mu \equiv \int x P(x)\, dx$$

$$\sigma^2 \equiv \int (x-\mu)^2 P(x)\, dx$$

Note that the variance is the average of the squared differences from the mean. This definition is chosen over something like the absolute value of the difference because the square is easier to calculate, giving us a convenient measure of the spread.

We are interested in the probability distribution function P(x). Now, you are trying to hit the center of the dartboard. Most of the darts will land close to the center. After all, you are aiming at the center. But occasionally you will hit some of the characters on the dartboard, like the one we have boxed off at dxdy.



The probability your dart will land in the dxdy patch is P(x)dx P(y)dy. We can use the same letter for each dimension because the functions will have the same form due to symmetry. The probability that a dart is at x or in the range up to x + dx is given by P(x)dx. Similarly for the range from y to y + dy we have P(y)dy. The probability to be in the patch then is the product of these probabilities.

Our probability to hit the cute little guy is given by

$$P(x)dx \cdot P(y)dy \, .$$

Since the probability to hit the circular ribbon area at the radial distance r from the center is independent of the angle, we can write

$$P(x)dx \cdot P(y)dy = g(r)dr$$, with

$$g(r) = P(x)P(y) \, .$$

Differentiating both sides with respect to θ brings us to

$$\frac{\partial g(r)}{\partial \theta} = 0 = \left[\frac{\partial P(x)}{\partial \theta}\right] P(y) + P(x)\left[\frac{\partial P(y)}{\partial \theta}\right]$$

Now, Cartesian coordinates are related to polar coordinates by

$$x = r\cos\theta \text{ and } y = r\sin\theta .$$

We proceed taking the derivatives using the chain rule.

$$0 = \left[\frac{dP(x)}{dx}\frac{\partial x}{\partial \theta}\right] P(y) + P(x)\left[\frac{dP(y)}{dy}\frac{\partial y}{\partial \theta}\right]$$

$$0 = \frac{dP(x)}{dx}(-r\sin\theta)P(y) + P(x)\frac{dP(y)}{dy} r\cos\theta$$

$$0 = -\frac{dP(x)}{dx} yP(y) + P(x)\frac{dP(y)}{dy} x$$

$$P'(x)yP(y) = P(x)P'(y)x$$

$$\frac{P'(x)}{xP(x)} = \frac{P'(y)}{yP(y)} = C$$

We have separated the variables here. Since the variables x and y are independent, the above must be equal to a constant. We want to solve this following differential equation.

$$\frac{P'(x)}{xP(x)} = C$$

$$\frac{1}{xP(x)}\frac{dP(x)}{dx}=C$$

$$\frac{1}{P(x)}dP(x)=Cxdx$$

$$\int\frac{1}{P(x)}dP(x)=\int Cxdx$$

$$\ln P(x)=\frac{Cx^2}{2}+c$$

$$P(x)=e^{\frac{Cx^2}{2}+c}$$

$$P(x)=e^{\frac{Cx^2}{2}}e^c$$

$$P(x)=Ae^{\frac{Cx^2}{2}}$$

Since the probability is less the farther the dart hits from the center,

$C=-k$, where $k>0$.

$$P(x)=Ae^{-\frac{k}{2}x^2}$$

There is your Gaussian, the classic bell or bell-shaped curve!

The total probability must be 1. Therefore,

$$\int_{-\infty}^{\infty} P(x)\,dx = A\int_{-\infty}^{\infty} e^{-\frac{k}{2}x^2}\,dx = 1$$

Now we use our result from before.

$$I = \int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx = \sqrt{\frac{\pi}{\alpha}}$$

We find

$$A\int_{-\infty}^{\infty} e^{-\frac{k}{2}x^2}\,dx = A\sqrt{\frac{2\pi}{k}} = 1$$

Therefore,

$$A = \sqrt{\frac{k}{2\pi}}$$

Our probability distribution is

$$P(x) = \sqrt{\frac{k}{2\pi}}e^{-\frac{k}{2}x^2}$$

What is k? Let's try to express k in terms of the dispersion.

$$\sigma^2 = \int (x-\mu)^2 P(x)\,dx$$

Note $\mu = 0$ due to the symmetry. Then our dispersion simplifies.

$$\sigma^2 = \int x^2 P(x)\,dx$$

$$\sigma^2 = \int x^2 P(x)\,dx = \int_{-\infty}^{\infty} x^2 \sqrt{\frac{k}{2\pi}} e^{-\frac{k}{2}x^2}\,dx$$

Now we use our derivative trick to do the integral where $\alpha = \frac{k}{2}$. It helps to keep in mind that $\frac{1}{\alpha} = \frac{2}{k}$.

$$I = \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2}\,dx = -\frac{d}{d\alpha}\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx = \frac{d}{d\alpha}\sqrt{\frac{\pi}{\alpha}} = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}$$

$$\sigma^2 = \sqrt{\frac{k}{2\pi}}\int_{-\infty}^{\infty} x^2 e^{-\frac{k}{2}x^2}\,dx = \sqrt{\frac{k}{2\pi}}\,\frac{1}{2}\,\frac{2}{k}\sqrt{\frac{2\pi}{k}}$$

$$\sigma^2 = \sqrt{\frac{k}{2\pi}}\int_{-\infty}^{\infty} x^2 e^{-\frac{k}{2}x^2}\,dx = \frac{1}{k}.$$

Then our

$$P(x) = \sqrt{\frac{k}{2\pi}} e^{-\frac{k}{2}x^2}$$

becomes

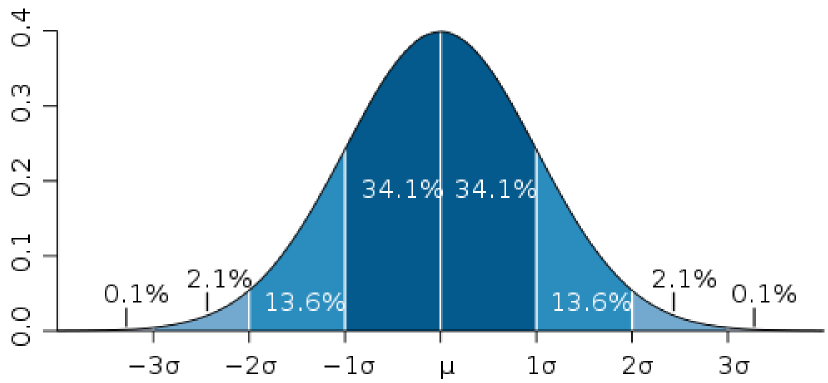
$$P(x) = \sqrt{\frac{1}{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}.$$

For an average $\mu \neq 0$, we use our shift trick. Remember how we shift a function?

$$\boxed{P(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$$

N3. The Gaussian in Statistics.

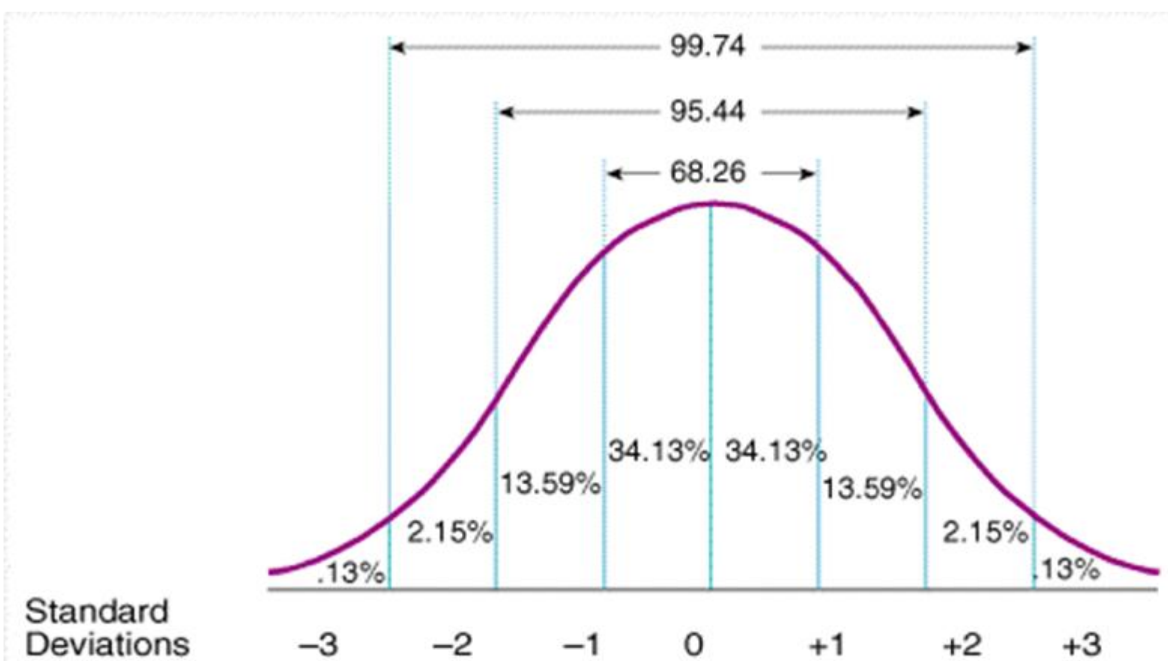
Gaussian Courtesy Petter Strandmark, Wikimedia



Measurements of any kind give Gaussians as the number of measurements gets large. The most likely value is μ and σ is called the standard deviation. Note the percentages for the area strips away from the center by σ, 2σ, 3σ.

In social science classes, it is practical to memorize these: about 68% for within plus or minus one standard deviation, 95% for within plus or minus 2σ and over 99% for within plus or minus 3σ. When you are more precise, using more significant figures and rounding off last, you have the results below.

Some remember this as the 68-95-99.7 rule.



Courtesy Hofstra

N4. A Delta Sequence of Gaussians.

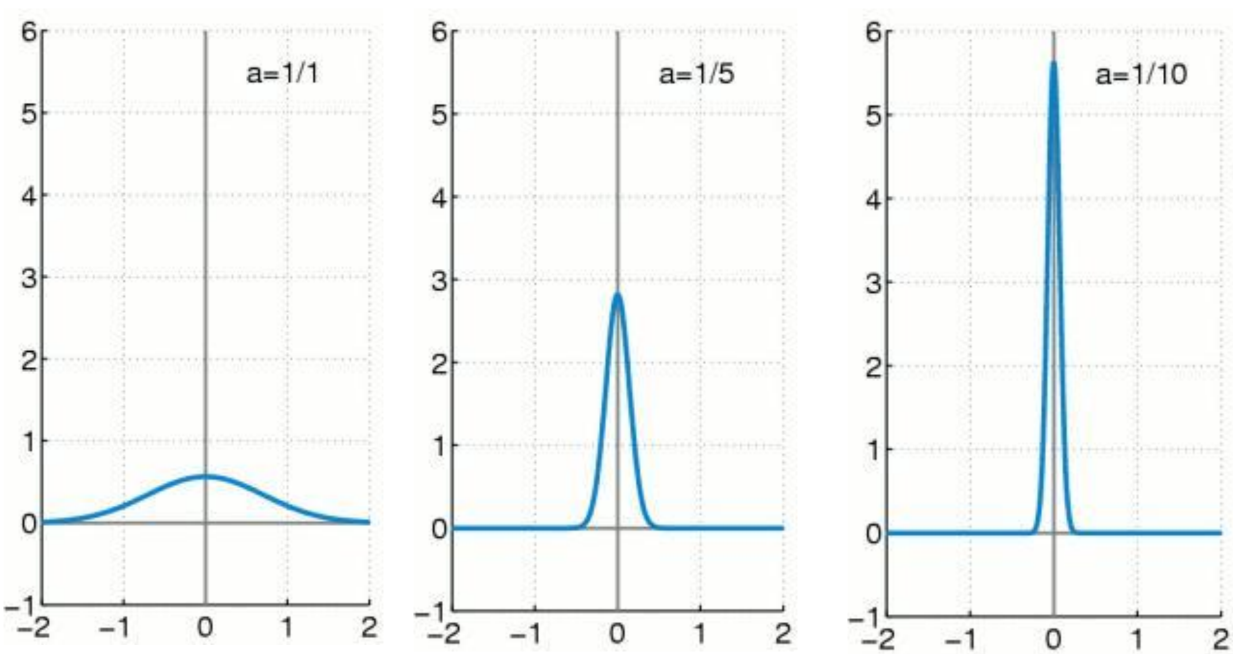
Now it is time to clean things up related to the delta function. We can arrive at a mathematically sound approach to the delta function by considering the delta function as a limit of a sequence of Gaussians. Let's return to our Gaussian probability function.

$$P(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

Here is a sequence example from Wikipedia. Let $a^2 = 2\sigma^2$. Then define

$$\delta_a(x) = \frac{1}{a\sqrt{\pi}} e^{-\frac{x^2}{a^2}} .$$

Check out the graphs for three values for *a* as *a* gets smaller and smaller.



Courtesy Wikipedia

So we write

$$\delta(x) = \lim_{a \to 0} \delta_a(x) = \lim_{a \to 0} \left[\frac{1}{a\sqrt{\pi}} e^{-\frac{x^2}{a^2}} \right]$$

or

$$\delta(x) = \lim_{\sigma \to 0} \delta_\sigma(x) = \lim_{\sigma \to 0} \left[\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \right]$$

The standard deviation σ heads to zero for the no-spread super-tall Dirac delta function.

"Magic Integral Form for the Delta Function." Our last trick is to show how the Dirac delta function can be written as an integral.

Remember this integral from the beginning of our course which we derived using the "completing-the-square" trick?

$$\int_{-\infty}^{\infty} e^{-\alpha x^2 + ikx}\, dx = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{k^2}{4\alpha}}$$

Here is the result if you swap x and k.

$$\int_{-\infty}^{\infty} e^{-\alpha k^2 + ikx}\, dk = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{x^2}{4\alpha}}$$

Let's make that right side look like a function in the delta sequence.

$$\delta_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

We start by making the assignment $4\alpha = 2\sigma^2$, i.e., $\alpha = \dfrac{\sigma^2}{2}$.

Then

$$\int_{-\infty}^{\infty} e^{-\alpha k^2 + ikx}\, dk = \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{x^2}{4\alpha}}$$

becomes

$$\int_{-\infty}^{\infty} e^{-\frac{\sigma^2 k^2}{2} + ikx}\, dk = \frac{\sqrt{2\pi}}{\sqrt{\sigma^2}}\, e^{-\frac{x^2}{2\sigma^2}}$$

We need to multiply by $\dfrac{1}{2\pi}$ to get a delta sequence on the right.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{\sigma^2 k^2}{2} + ikx}\, dk = \frac{1}{2\pi}\frac{\sqrt{2\pi}}{\sqrt{\sigma^2}}\, e^{-\frac{x^2}{2\sigma^2}}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{\sigma^2 k^2}{2} + ikx}\, dk = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{x^2}{2\sigma^2}} = \delta_\sigma(x)$$

Watch this trick.

$$\delta(x) = \lim_{\sigma \to 0} \delta_\sigma(x) = \frac{1}{2\pi} \lim_{\sigma \to 0} \int_{-\infty}^{\infty} e^{-\frac{\sigma^2 k^2}{2} + ikx}\, dk$$

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\, dk$$

What is this? You can't integrate cosines and sines to get a delta function.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk = \frac{1}{2\pi}\frac{1}{ix}e^{ikx}\Big|_{k=-\infty}^{k=\infty} = ?$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk = \frac{1}{2\pi}\frac{1}{ix}\left[\cos kx + i\sin kx\right]_{-\infty}^{\infty} = ?$$

Well, technically, we interpret all this with mathematical rigor by understanding there is a Gaussian in the integral and we are looking at a very narrow peaked Gaussian.

So we really mean take the limiting case of the following.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{\sigma^2 k^2}{2}+ikx}\,dk = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}} = \delta_\sigma(x)$$

With this understood, we don't worry about it anymore and present you with the following integral representation of the delta function.

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk$$

We will return to the delta function when we discuss Fourier transforms. This section has given you a foundation for understanding Fourier transforms, which we take up later in our course.