Theoretical Physics
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Chapter N Homework. The Dirac Delta Function

HW-N1. Probability Distribution and Moments. The n[th] central moment for the probability distribution P(x) is defined as

$$E[(x-\mu)^n] \equiv \int_{-\infty}^{\infty} (x-\mu)^n P(x)dx .$$

$$P(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$. Therefore $\mu = 0$ since P(x) is centered on zero.

$$E[(x-\mu)^n] = E[x^n] = \int_{-\infty}^{\infty} x^n P(x)dx$$

0[th] Moment.

$$E[x^0] = \int_{-\infty}^{\infty} x^0 P(x)dx = \int_{-\infty}^{\infty} P(x)dx = 1$$

1[st] Moment.

$$E[x^1] = \int_{-\infty}^{\infty} xP(x)dx = 0$$ since integrating odd function over symmetric region.

2[nd] Moment. $$E[x^2] = \int_{-\infty}^{\infty} x^2 P(x)dx = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx$$

We need $\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = -\frac{d}{d\alpha}\sqrt{\frac{\pi}{\alpha}} = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\frac{1}{\alpha}$ with $\alpha = \frac{1}{2\sigma^2}$.

$$E[x^2] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{2} \sqrt{\frac{2\sigma^2\pi}{1}} \frac{2\sigma^2}{1} = \sigma^2 ,$$

$$\boxed{E[x^0] = 1} \qquad \boxed{E[x^1] = 0} \qquad \boxed{E[x^2] = \sigma^2}$$

HW-N2. Probability Distribution and Moments Continued.

$$E[(x-\mu)^n] \equiv \int_{-\infty}^{\infty} (x-\mu)^n P(x)dx \; .$$

$$P(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$. Therefore $\mu = 0$ since P(x) is centered on zero.

$$E[(x-\mu)^n] = E[x^n] = \int_{-\infty}^{\infty} x^n P(x)dx$$

3rd Moment

$$E[x^3] = \int_{-\infty}^{\infty} x^3 P(x)dx = 0$$ since integrating odd function over symmetric region.

$$\boxed{E[x^3] = 0}$$

4th Moment.

$$E[x^4] = \int_{-\infty}^{\infty} x^4 P(x)dx = \int_{-\infty}^{\infty} x^4 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx$$

We need $\int_{-\infty}^{\infty} x^4 e^{-\alpha x^2} dx = -\frac{d}{d\alpha}\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = -\frac{d}{d\alpha}\left[\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\frac{1}{\alpha}\right]$

$$= -\frac{d}{d\alpha}\left[\frac{\sqrt{\pi}}{2} a^{-3/2}\right] = \frac{3}{4}\sqrt{\pi}\, a^{-5/2} = \frac{3}{4}\sqrt{\frac{\pi}{\alpha}}\frac{1}{\alpha^2}$$ with $\alpha = \frac{1}{2\sigma^2}$.

$$E[x^4] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^4 e^{-\frac{x^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{3}{4} \sqrt{\frac{2\sigma^2\pi}{1}} \frac{\left[2\sigma^2\right]^2}{1} = 3\sigma^4 \; .$$

$$\boxed{E[x^4] = 3\sigma^4}$$

HW-N3. Dirac Delta Function. Evaluate the following two integrals, showing all steps.

$$I_{k>0} = \int_{-\infty}^{\infty} f(x)\,\delta(kx)\,dx \text{ , where } k > 0$$

$$I_{k<0} = \int_{-\infty}^{\infty} f(x)\,\delta(kx)\,dx \text{ , where } k < 0$$

Hint: Let z = kx and use what you know about the delta function from class.

$$I_{k>0} = \int_{-\infty}^{\infty} f(\frac{z}{k})\,\delta(z)\,\frac{dz}{k} = \frac{1}{k}\int_{-\infty}^{\infty} g(z)\,\delta(z)\,dz = \frac{g(0)}{k} = \frac{f(0)}{k}$$

$$I_{k<0} = \int_{\infty}^{-\infty} f(\frac{z}{k})\,\delta(z)\,\frac{dz}{k} = -\int_{-\infty}^{\infty} f(\frac{z}{k})\,\delta(z)\,\frac{dz}{k} = \frac{f(0)}{-k}$$

Both cases can be combined as follows.

$$\boxed{\int_{-\infty}^{\infty} f(x)\,\delta(kx)\,dx = \frac{f(0)}{|k|}}$$