Theoretical Physics
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Chapter P Notes. Fourier Transforms

P1. Fourier Series with Exponentials.

Here is a summary of our last chapter, where we express a periodic wave as a Fourier series.

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left[a_m \cos(mx) + b_m \sin(mx) \right]$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin(nx) dx$$

Our goal is to replace the above series with an integral so we will be able to represent functions which are not periodic. Think of this as including sine waves with frequencies that fall in between the harmonic frequencies. But first we need to expand the interval. We will achieve our goal in four steps.

Our Four Steps

1. Introducing Exponentials.

2. Expanding the Interval.

3. Transforming to an Integral.

4. Aiming for Infinity.

We use the Euler relation

$$e^{ix} = \cos x + i \sin x$$

We can write cosines and sines in terms of exponentials.

PP1 (Practice Problem). Refresh your memory of working with Euler's relation to be sure you can easily arrive at these "backward" Euler relations.

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \text{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i} .$$

Using these, we can write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

PP2 (Practice Problem). Show the following.

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2}(a_n - ib_n) \quad \text{for} \quad n > 0$$

$$c_n = \frac{1}{2}(a_n + ib_n) \quad \text{for} \quad n < 0$$

We can calculate the "c" coefficients from our familiar formulas:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x)\,dx , \quad a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos(nx)\,dx , \quad \text{and}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin(nx)\,dx .$$

The results are worked out below.

The equations $c_0 = \frac{a_0}{2}$ and $a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\,dx$ lead to the following.

$$\boxed{c_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\,dx}$$

Our equation $c_n = \frac{1}{2}(a_n - ib_n)$ for $n > 0$ with

$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos(nx)\,dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin(nx)\,dx$ gives

$c_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\left[\cos(nx) - i\sin(nx)\right]dx$ for $n > 0$.

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\,e^{-inx}\,dx$$

The equation $c_n = \frac{1}{2}(a_n + ib_n)$ for $n < 0$ is

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\,e^{inx}\,dx$$

All of these can be written for all n as follows.

$$\boxed{c_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\,e^{-inx}\,dx}$$

Summary

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x) e^{-inx} dx$$

P2. Expanding the Interval. For notational purposes, rewrite the above with a new z-variable and g(z).

$$g(z) = \sum_{n=-\infty}^{\infty} c_n e^{inz}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} g(z) e^{-inz} dz$$

Then expand the interval by a transformation of variables.

$$-\pi \le z \le +\pi$$

$$-L \le x \le +L$$

This leads us to $\frac{z}{x} = \frac{\pi}{L}$, i.e., $z = \frac{\pi}{L}x$ and $dz = \frac{\pi}{L}dx$. We arrive at

$$g(\frac{\pi}{L}x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{L}x}$$ and $$c_n = \frac{1}{2\pi}\int_{-L}^{+L} g(\frac{\pi}{L}x) e^{-i\frac{n\pi}{L}x} \frac{\pi}{L} dx.$$

We can write these as follows.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{L}x}$$

$$c_n = \frac{1}{2L}\int_{-L}^{+L} f(x) e^{-i\frac{n\pi}{L}x} dx$$

P3. Transforming to an Integral.

It is time for some "theoretical physics" magic. Note that this chapter is not meant to be super mathematically rigorous. Our focus here is trying to understand where the Fourier transform comes from rather than just giving it to you.

We would like to transform the series to an integral. We note that since n are integers, then $\Delta n = 1$. We then write

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{L}x} \Delta n \ .$$

Now we introduce a new variable, one which we intend to promote to a continuous variable.

$$k = \frac{n\pi}{L} \ .$$

Therefore,

$$\Delta k = \frac{\pi}{L}\Delta n \quad \text{and} \quad \Delta n = \frac{L}{\pi}\Delta k \ .$$

Let $Lc_n \to c(k)$. With the new variable k

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} c(k) e^{ikx} dk \ .$$

The variable k has become a continuous variable and we have replaced the sum with an integral. The three things we did: 1) replace delta k with dk, 2) "rip off" the n from the c and introduce c(k), and 3) turn the summation sign into a "snake" where we integrate over all k since our sum did that for the discrete case.

What about our other equation?

$$c_n = \frac{1}{2L}\int_{-L}^{+L} f(x)e^{-i\frac{n\pi}{L}x}dx$$

With our new variable we have

$$Lc_n \to c(k) = \frac{1}{2}\int_{-L}^{+L} f(x)e^{-ikx}dx \; .$$

Summary:

$$\boxed{f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} c(k)e^{ikx}dk}$$

$$\boxed{c(k) = \frac{1}{2}\int_{-L}^{+L} f(x)e^{-ikx}dx}$$

P4. Aiming for Infinity.

Now comes a "questionable" mathematical step. Can we extend the limits of integration to infinity?

$$c(k) = \frac{1}{2}\int_{-\infty}^{+\infty} f(x)e^{-ikx}dx$$

Doesn't $c(k) = Lc_n \to \infty$ and we get nonsense? But what about c_n in this limit?

$$c_n = \frac{c(k)}{L} \to 0$$

For now, let's hope $c(k)$ is finite and the write the following pair of equations.

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} c(k)e^{ikx}dk \quad \text{and} \quad c(k) = \frac{1}{2}\int_{-\infty}^{+\infty} f(x)e^{-ikx}dx$$

Well, if we substitute $c(k)$ in the $f(x)$ integral using some x' for the integration to get the $c(k)$, we should get $f(x)$ back again. It is important to use something like x' because we have an "x" on the left side.

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} c(k)e^{ikx}dk$$

Now substitute

$$c(k) = \frac{1}{2}\int_{-\infty}^{+\infty} f(x')e^{-ikx'}dx'$$

and see what we get.

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{1}{2}\int_{-\infty}^{+\infty} f(x')e^{-ikx'}dx'\right]e^{ikx}dk$$

We are going to do the k integral first. Note x' and k are independent of each other.

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{1}{2}\int_{-\infty}^{+\infty} f(x')e^{-ikx'}e^{ikx}dx'\right]dk$$

$$f(x) = \int_{-\infty}^{\infty} f(x')\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ik(x-x')}dk\right]dx'$$

Do you see it? Do you recognize what is in the brackets? Here is a reminder below.

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}dk \quad \text{and} \quad \delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ik(x-x')}dk$$

This means we have

$$f(x) = \int_{-\infty}^{\infty} f(x')\delta(x-x')\,dx',$$

which is a true statement. Everything checks out.

Summary. The following are consistent.

$$\boxed{f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} c(k)e^{ikx}dk}$$

$$\boxed{c(k) = \frac{1}{2}\int_{-\infty}^{+\infty} f(x)e^{-ikx}dx}$$

P5. The Fourier Transform. We will the following convention for defining the Fourier Transform. Our convention will involve a symmetric definition, but you do not have to do this. Some authors go have 2π as a unit. The convention we will use to define

$$c(k) = \sqrt{\frac{\pi}{2}} F(k) .$$

Then, our two equations

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} c(k) e^{ikx} dk \quad \text{and} \quad c(k) = \frac{1}{2}\int_{-\infty}^{+\infty} f(x) e^{-ikx} dx$$

become

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} c(k) e^{ikx} dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k) e^{ikx} dk$$

$$\sqrt{\frac{\pi}{2}} F(k) = \frac{1}{2}\int_{-\infty}^{+\infty} f(x) e^{-ikx} dx$$

The result is the symmetric equations below.

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k) e^{ikx} dk}$$

$$\boxed{F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x) e^{-ikx} dx}$$

The function $F(k)$ is called the Fourier transform of $f(x)$ and $f(x)$ is the inverse Fourier transform of $F(k)$.

Some authors write the Fourier transform with the following notation.

$$\Im\{f(x)\} \equiv F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx$$

The inverse Fourier transform is then

$$\Im^{-1}\{F(k)\} \equiv f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx} dk$$

P6. Parseval's Theorem. In quantum mechanics the probability distribution is given by

$$P(x) = f^*(x) f(x)$$

In terms of the Fourier transform,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx} dk \quad \text{and}$$

$$f^*(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F^*(k') e^{-ik'x} dk'$$

Note our careful use of two integration variables k and k' since we plan on multiplying these together. Remember, the k and k' are integration variables similar to our summation variations we encountered earlier.

$$P(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F^*(k') e^{-ik'x} dk' \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{-ikx} dk$$

$$P(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^*(k') F(k) e^{i(k-k')x} dk\, dk'$$

Let's integrate the probability distribution over all x.

$$\int_{-\infty}^{\infty} P(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} P(x)\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F^*(k')F(k)e^{i(k-k')x}dk\,dk'dx$$

Now, watch this simplification. Rearrange things and plan on doing the x integration first.

$$\int_{-\infty}^{\infty} P(x)\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F^*(k')F(k)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(k-k')x}dx\right]dkdk'$$

Note that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(k-k')x}dx = \delta(k-k') \, .$$

Then

$$\int_{-\infty}^{\infty} P(x)\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F^*(k')F(k)\delta(k-k')dkdk' \, ,$$

which means we can do one of the remaining two integrals quickly due to the delta function.

$$\int_{-\infty}^{\infty} P(x)\,dx = \int_{-\infty}^{\infty} F^*(k)F(k)dk$$

But this tells us

$$\boxed{\int_{-\infty}^{\infty} f^*(x)f(x)\,dx = \int_{-\infty}^{\infty} F^*(k)F(k)dk}$$

This is Parseval's Theorem.

If $f(x)$ is normalized, so is $F(k)$.