Theoretical Physics
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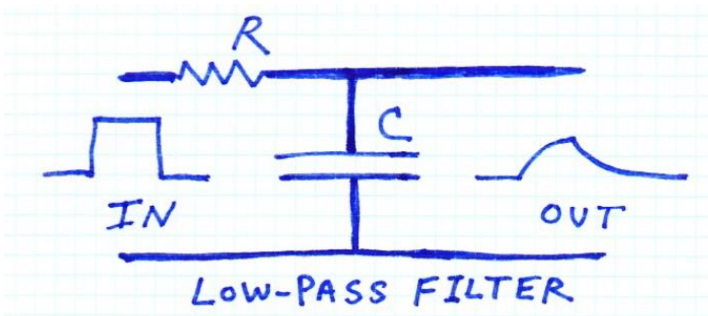
Chapter U Notes. Green's Functions

U1. Impulse Response. We return to our low-pass filter with $R=1$, $C=1$, and $f(t)=1$ for 1 second from $t=0$ to $t=1$. The initial charge on the capacitor is $q(0)=0$. We have already solved this problem.



The charge on the capacitor is given by the following integral.

$$q(t)=\int_0^t f(u)g(t-u)du$$

This solution is the convolution of our input voltage function $f(t)$ and the capacitor-decay response function $g(t)=e^{-t}$. The input function is for us to choose, but the capacitor response function is intrinsic to our low-pass filter.

Our differential equation is $f(t)=V=IR+\frac{q}{C}=I+q=\frac{dq}{dt}+q$. The Laplace transform gives

$$F(s)=sQ(s)+Q(s)$$

$$Q(s)=F(s)\frac{1}{s+1}=F(s)G(s)$$

We know from before that a product in Laplace-transform s-space means a convolution of the respective functions in our t-space. The s-space shows clearly the separation of our input function and the response function. All of the secrets of the system are in $G(s)$. What is the formal way to solve for this without the $F(s)$ getting in the way? Well, if you set $f(t)=0$, that kills things because the Laplace transform of zero is zero and you would get $Q(s)=(0)\left[\frac{1}{s+1}\right]=(0)G(s)=0$.

So we really want

$$Q(s) = F(s)G(s) = (1)G(s) = G(s)$$

in order to get to know the system for itself. So we are looking for a function $f(t)$ that has a Laplace transform of 1.

$$L\{f(t)\} = F(s) = 1$$

Therefore, we want

$$L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = 1 .$$

Do you recognize it? The Dirac delta function comes to the rescue.

$$L\{\delta(t)\} = \int_0^\infty \delta(t)e^{-st}dt = e^{-st}\Big|_{t=0} = 1$$

Then, the solution $q(t)$ is given by the inverse transform of $G(s)$. We get the essential response of the circuit to hitting the system with the Dirac delta function. This kind of input is called an impulse and the solution is the impulse response.

$$Q(s) = G(s) = \frac{1}{s+1}$$

The impulse solution is our $g(t)$ function

$$g(t) = e^{-t}$$

This special function is called the Green's function. Then, our general solution for some arbitrary input function $f(t)$ is given by the convolution $f(t) * g(t)$.

$$q(t) = \int_0^t f(u)g(t-u)du$$

Note that each decay response $g(t-u)$ kicks in at $t = u$. Each voltage jolt $f(u)$ at time u initiates a shifted decay response $g(t-u) = e^{-(t-u)}$. It all makes sense!

U2. The Green's Function.

George Green (1793-1841)
Source: www.dert2007.org.uk
Nottingham People

Consider some system without any external input or interference. Let this system be described by a differential equation where we let D be the differential operator for the system.

$$D\{x(t)\} = 0$$

For our example $IR + \frac{q}{C} = 0$, we have

$$D = R\frac{d}{dt} + \frac{1}{C} \text{ and } x(t) = q(t).$$

Note that in our original differential equation $IR + \frac{q}{C} = V = f(t)$, we remove the input voltage function. We choose $V = f(t) = 0$. We want to isolate the system from external driving forces and input activity. Now we hit the system with a Delta function.

$$D\{x(t)\} = \delta(t)$$

Our solution will be the Green's function, i.e., the impulse response function.

$$D\{g(t)\} = \delta(t)$$

Then, the general solution to an arbitrary input or driving force:

$$D\{x(t)\} = f(t)$$

will be the convolution of $f(t)$ with $g(t)$: $x(t) = \int_0^t f(u)g(t-u)du$

We will write the Green's function $g(t-u)$ with the following notation.

$$G(t,u) \equiv g(t-u)$$

Then, the convolution equation becomes $x(t)=\int_0^t f(u)G(t,u)du$. It is more intuitive to write the Green's function first in the integrand.

$$x(t)=\int_0^t G(t,u)\,f(u)du$$

Here is how we can interpret this amazing integral. The Green's function takes the input driving function $f(u)$ and works on each infinitesimal action, applying the impulse response intrinsic to the system at the appropriate time-shift $t-u$.

Remember that with no time shift,

$$G(t,u)=g(t-u) \text{ becomes } G(t,0)=g(t).$$

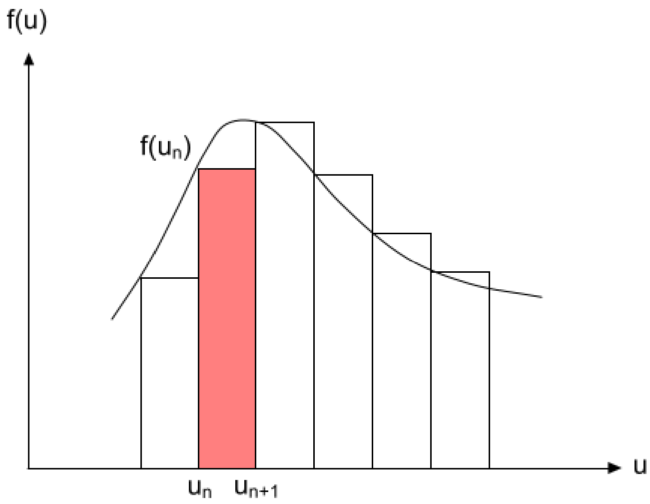
This is our basic response to the Dirac-delta function impulse

$$G(t,0)=g(t)$$

So the time-shift comes in to adjust for each input action at time u. The shifted response is $G(t,u)=g(t-u)$. So the Green's function has the convolution built into it. The Green's function works on all the input segments and tells us how the system responds to the arbitrary input function. It connects u to t, applying the appropriate time shift. Read the integral below from right to left and you go from u to t.

$$x(t)=\int_0^t G(t,u)\,f(u)\,du$$

Though we are using modern notation and work of Dirac that came later, the Green's function dates back to the 1800s to George Green. What is even more amazing is that George Green, the son of a baker, was mostly self-taught as a mathematical physicist. He finally went to college at age 40. Not long after graduation Green became ill and died at age 47 without realizing how important his work would become. We have studied two major contributions: Green's Theorem and Green's Functions.



Finally, consider the f(u) function as a series of discrete impulse strips. For each strip we need to apply the shifted impulse response G(t,u).

$$x(t) = \sum_n G(t,u_n)\, f(u_n)\Delta u$$

The Green's function takes care of this for us.

$$x(t) = \int_0^t G(t,u)\, f(u)\, du$$

U3. Fourier Transform Space.

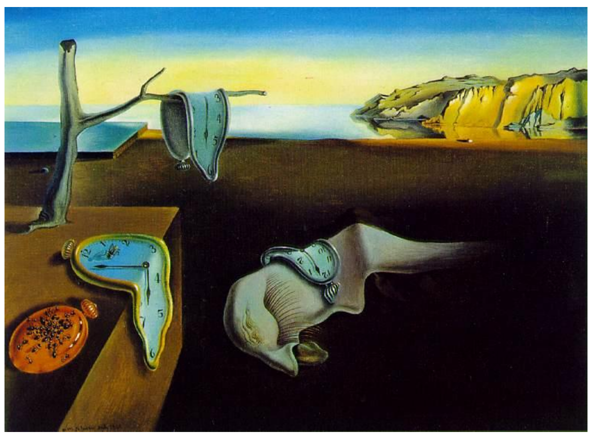
Why a return to transforms? With our knowledge of complex integration, we are going to calculate a Green's function by transforming a differential equation to Fourier transform space, solving the algebraic equation, and getting back on our own to regular space without the use of a table!

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt$$

Time Space "t"

Geneva Clock Courtesy amazon.com

Fourier Transform "ω" Space



The Persistence of Memory (1931)
by Salvador Dali (1904-1989)
Courtesy www.artchive.com

First we need the Fourier transform of derivatives. Do you remember why the minus sign in the Fourier transform according to our convention?

$$\Im\{\frac{df(t)}{dt}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{df(t)}{dt} e^{-i\omega t} dt$$

We use integration by parts now as we did for the Laplace transform case.

$$\frac{d}{dt}\left[f(t)e^{-i\omega t} \right] = \frac{df(t)}{dt} e^{-i\omega t} - i\omega f(t)e^{-i\omega t}$$

$$\Im\{\frac{df(t)}{dt}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d}{dt}\left[f(t)e^{-i\omega t} \right] dt + \frac{1}{\sqrt{2\pi}} i\omega \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt$$

$$\Im\{\frac{df(t)}{dt}\} = \frac{1}{\sqrt{2\pi}} f(t)e^{-i\omega t}\Big|_{-\infty}^{\infty} + i\omega F(\omega)$$

Remember our functions f(t) must be well-behaved and vanish at infinity for us to even have a Fourier transform. Therefore for the derivative we get the following.

$$\Im\{\frac{df(t)}{dt}\} = i\omega F(\omega)$$

Note that the boundary conditions at plus and minus infinity replace the initial condition appearing in the Laplace transform. For the second derivative we let $g(t) = f'(t)$.

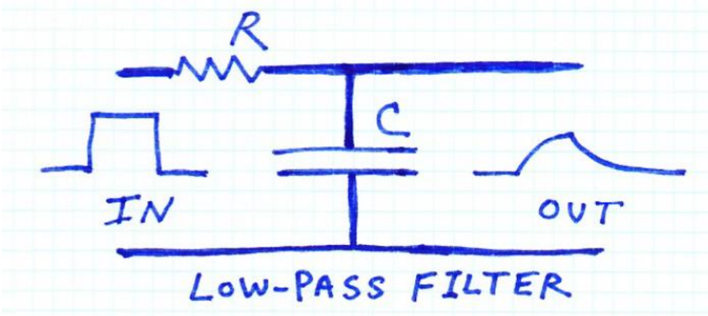
$$\Im\{\frac{d^2 f(t)}{dt^2}\} = \Im\{\frac{dg(t)}{dt^2}\} = i\omega G(\omega) = i\omega \Im\{\frac{df(t)}{dt}\} = -\omega^2 F(\omega)$$

Summary - derivatives melt away in Fourier transform space just as they do in Laplace transform space.

$$\Im\{\frac{df(t)}{dt}\} = i\omega F(\omega) \qquad \Im\{\frac{d^2 f(t)}{dt^2}\} = -\omega^2 F(\omega)$$

U4. The Green's Function for the Low-Pass Filter. We return to our low-pass filter with $R=1$, $C=1$, and $V_{in} = f(t)$. The differential equation is

$$V_{in} = IR + \frac{q}{C} = R\frac{dq}{dt} + \frac{q}{C} = \frac{dq}{dt} + q .$$



We know the answer:

$$q(t) = \int_0^t G(t,u)\, f(u)du$$

where $G(t,u)$ is the shifted Green's function $G(t,0) = e^{-t}$, i.e., $G(t,u) = e^{-(t-u)}$. We are going to solve for the Green's function using the sophisticated tools we have developed. This serves as a unifying example whereby we see a relationship among the Dirac delta function, Fourier transforms, complex integration, and the Green's function. So it also serves as an excellent review!

Step 1. The Dirac Delta Function. You set up your differential equation with a Dirac-delta-function impulse. In other words, at time $t=0$ we smack the system with an impulse voltage à la Dirac delta.

$$\frac{dq}{dt} + q = \delta(t)$$

Step 2. The Fourier Transform. You take the Fourier transform of the differential equation to transform it into an algebraic one in transform space.

$$\Im\{\frac{dq}{dt} + q\} = \Im\{\delta(t)\} \quad \text{with} \quad \Im\{q(t)\} = Q(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} q(t) e^{-i\omega t} dt$$

Then $$i\omega Q(\omega) + Q(\omega) = \frac{1}{\sqrt{2\pi}}$$ and $$Q(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{1+i\omega} .$$

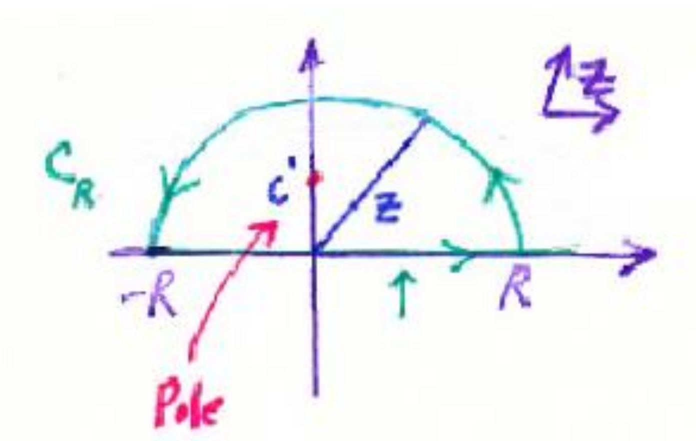
Step 3. Complex Integration. You do an inverse Fourier transform to get back to your regular "t" space. You will need complex integration as your "key." Instead of looking up your "key" in a table as we did for the Laplace transform, you will fashion the "key" yourself. This is your "key" to your portal to get back to "t" space.

Take the inverse Fourier Transform of this function: $Q(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{1+i\omega}$.

Below is our inverse Fourier transform. Remember that the "+" goes on the exponential when you fashion your f(t) from exponentials. This is our convention for the transforms.

$$\Im^{-1}\{Q(\omega)\} = q(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Q(\omega) e^{i\omega t} d\omega$$

$$q(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{1+i\omega} e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+i\omega} e^{i\omega t} d\omega$$



$$q(t) = \frac{1}{2\pi} \oint \frac{1}{1+iz} e^{izt} dz$$

Multiply top and bottom by $-i$.

$$q(t) = \frac{-i}{2\pi} \oint \frac{1}{z-i} e^{izt} dz$$

We have one pole and $t > 0$ is our time of interest. This means along the vertical imaginary axis where $z = iR$, we have $e^{i(iR)t} = e^{-Rt}$, which goes to zero as the radius goes to infinity as needed. We now need this e^{-tR} factor since we no longer have a $1/z^2$ in the denominator.

$$q(t) = \frac{-i}{2\pi} 2\pi i Res(\frac{e^{izt}}{z-i}) = e^{izt}\Big|_{z=i} = e^{-t}$$

Comment on Needing e^{izt}. Let $z = \text{Re}^{i\theta}$ as we did earlier in our course.

$$I_C = \frac{1}{2\pi}\int_{C_R} \frac{1}{1+iz} e^{izt} dz = \frac{1}{2\pi}\int_0^{\pi} \frac{1}{1+i\,\text{Re}^{i\theta}} e^{iR(\cos\theta + i\sin\theta)t} i\,\text{Re}^{i\theta}$$

$$I_C = \frac{1}{2\pi}\int_0^{\pi} \frac{1}{1+i\,\text{Re}^{i\theta}} e^{-R(\sin\theta)t} e^{iR(\cos\theta)t} i\,\text{Re}^{i\theta} d\theta$$

$$I_C \underset{large\ R}{\rightarrow} \frac{1}{2\pi}\int_0^{\pi} \frac{1}{\text{Re}^{i\theta}} e^{-R(\sin\theta)t} e^{iR(\cos\theta)t}\,\text{Re}^{i\theta} d\theta$$

$$I_C \rightarrow \frac{1}{2\pi}\int_0^{\pi} e^{-R(\sin\theta)t} e^{iR(\cos\theta)t} d\theta$$

Now you see that the $e^{-R(\sin\theta)t}$ saves us. What about when $\sin\theta = 0$ at the end points. No problem! We want those points in the x-axis integration. So we really need to integrate from an infinitesimal displaced point. And once we leave the x-axis with a finite angle, the infinite R kicks in and we are dead.

Step 4. Green's Function. You have your Green's function.

$$G(t,0) = e^{-t}$$

For the general solution for some arbitrary $f(t)$, you apply your time-shifted Green's function

$$G(t,u) = e^{-(t-u)}$$

and write

$$q(t) = \int_0^t G(t,u) f(u) du\ .$$

Some authors prefer the following notation.

$$q(t) = \int_0^t G(t,t')f(t')dt' \quad \text{with} \quad G(t,t') = e^{-(t-t')}$$

The primed time is the past time for which an external voltage was interacting with our system and "t" itself with no prime is the current time.

Below is a summary of our journey.