Theoretical Physics
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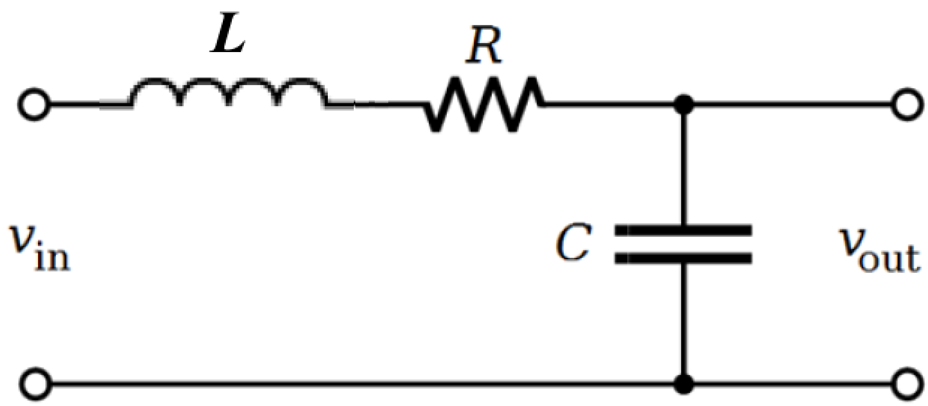
Chapter V Homework - Solutions. Transfer Functions.

V1. The LRC Circuit. Find the transfer function $H(\omega)$ for the LRC circuit shown below.



$$H(\omega) = \frac{V_{out}(t)}{V_{in}(t)} = \frac{1/(j\omega C)}{j\omega L + R + \dfrac{1}{j\omega C}}$$

$$H(\omega) = \frac{V_{out}(t)}{V_{in}(t)} = \frac{1}{j\omega L j\omega C + j\omega RC + 1}$$

$$H(\omega) = \frac{V_{out}(t)}{V_{in}(t)} = \frac{1}{-\omega^2 LC + j\omega RC + 1}$$

$\omega_0 = \dfrac{1}{\sqrt{LC}}$ and $\beta = \dfrac{R}{2L}$ so that $j\omega RC = j\omega \dfrac{R}{2L} 2LC = j\omega\beta \dfrac{2}{\omega_0^2}$

$$H(\omega) = \frac{1}{1 - \frac{\omega^2}{\omega_0^2} + 2\beta \frac{\omega}{\omega_0^2} j} \qquad H(\omega) = \frac{\omega_0^2}{\omega_0^2 - \omega^2 + 2\beta\omega\, j}$$

$$\left|H(\omega)\right| = \frac{\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$ since $$\left|\frac{1}{z}\right| = \frac{1}{|z|}$$

From $H(\omega) = \dfrac{\omega_0^2}{\omega_0^2 - \omega^2 + 2\beta\omega\, j}$, we get

$$H(\omega) = \frac{\omega_0^2(\omega_0^2 - \omega^2 - 2\beta\omega\, j)}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$ and $$\tan\phi = \frac{-2\beta\omega}{\omega_0^2 - \omega^2}$$

Summary

$$\left|H(\omega)\right| = \frac{\omega_0^2}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\beta^2\omega^2}} \qquad \tan\phi = \frac{2\beta\omega}{\omega^2 - \omega_0^2}$$

$$V_{in}(t) = V_0 \sin\omega t \qquad V_{out}(t) = \left|H(\omega)\right| V_0 \sin(\omega t + \phi)$$

$$q(t) = CV_{out}$$

$$q(t) = C\frac{\omega_0^2}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\beta^2\omega^2}} V_0 \sin(\omega t + \phi)$$

$$q(t) = \frac{V_0}{L}\frac{\sin(\omega t + \phi)}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\beta^2\omega^2}}$$

Sketch $A(\omega)$ that appears in $q(t) = A(\omega)\sin(\omega t + \phi)$, i.e.,

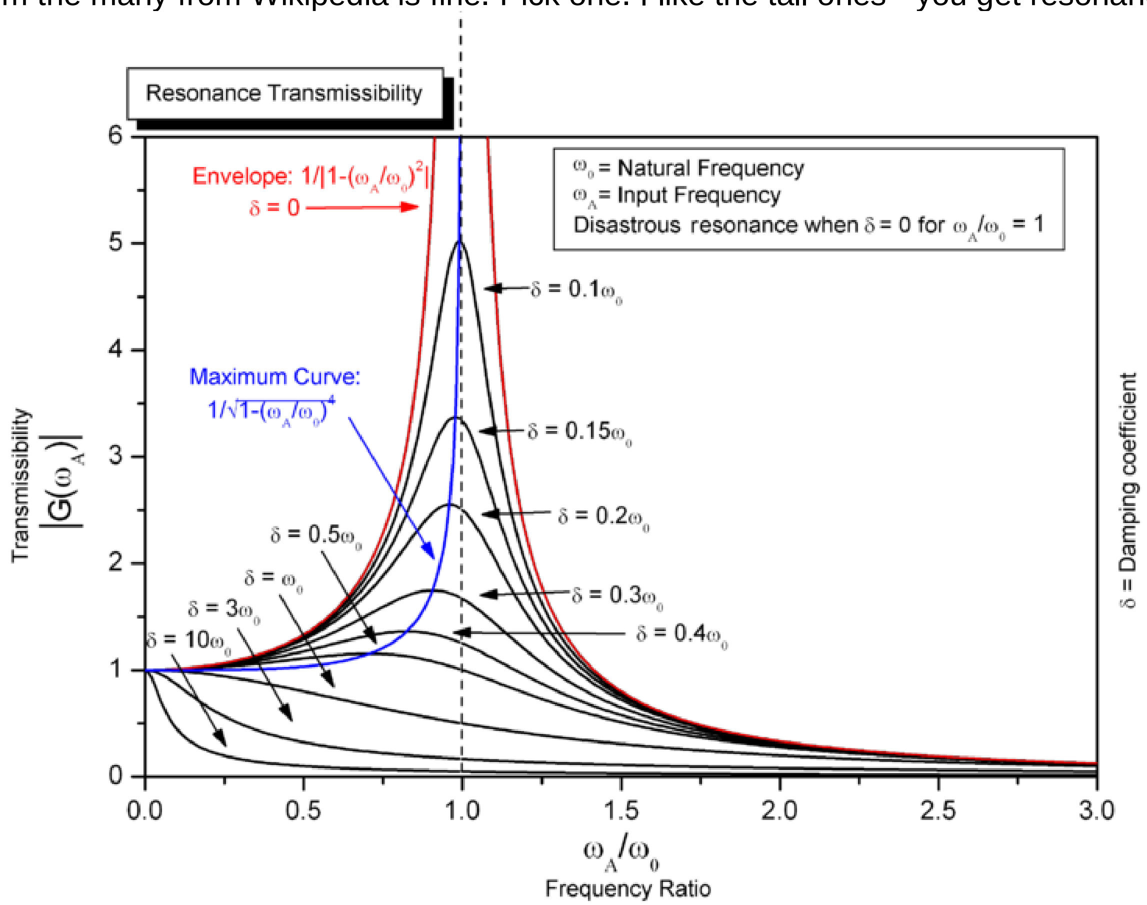
$$A(\omega) = \frac{CV_0\omega_0^2}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\beta^2\omega^2}}$$ NOTE $\omega \geq 0$ (a frequency).

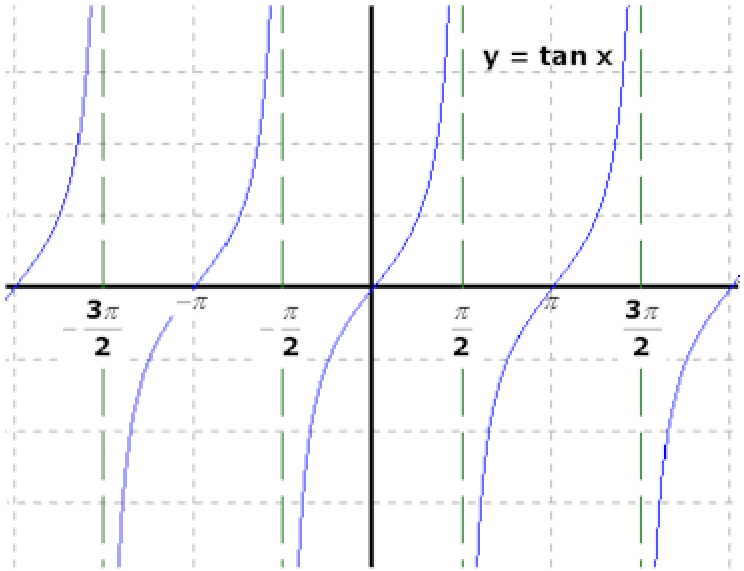
Analysis: $$A(0) = \frac{CV_0\omega_0^2}{\sqrt{(0^2 - \omega_0^2)^2 + 4\beta^2 0^2}} = \frac{CV_0\omega_0^2}{\omega_0^2} = CV_0$$

$$\lim_{\omega\to\infty} A(\omega) = "\frac{CV_0\omega_0^2}{\sqrt{(\infty^2 - \omega_0^2)^2 + 4\beta^2\infty^2}}" = 0$$

$$A(\omega_0) = \frac{CV_0\omega_0^2}{\sqrt{(\omega_0^2 - \omega_0^2)^2 + 4\beta^2\omega_0^2}} = \frac{CV_0\omega_0^2}{2\beta\omega_0} = \frac{CV_0\omega_0}{2\beta}$$

Since the values will determine the intermediate heights, we are looking for your sketch to start at some constant and then go to zero at infinity. Any of the following sketches from the many from Wikipedia is fine. Pick one. I like the tall ones - you get resonance!





Sketch

$$\phi(\omega) = \tan^{-1}\frac{2\beta\omega}{\omega^2 - \omega_0^2}$$

First, use the reference from UBC Wiki at the left for the tan(x). This will help us.

Analysis:

$$\phi(0) = \tan^{-1}(0) = 0$$

When the frequency starts to increase, it is at first $\omega < \omega_0$, which means we are taking the arctangent of a negative number. As $\omega \to \omega_0$ we have the arctangent of negative infinity from the lesser side. This means our phase angles heads to $-\frac{\pi}{2}$. Then when $\omega_0 \leftarrow \omega$ from the right side we have $\omega \to \infty$ and we have the arctangent $-\frac{\pi}{2}$ again, but this time look at the upper blue curve at $-\frac{\pi}{2}$. Then, we have the arctangent of positive "stuff" when $\omega_0 < \omega$. So we slide down this blue curve with phase angle going from $-\frac{\pi}{2}$ to $-\pi$. Note that

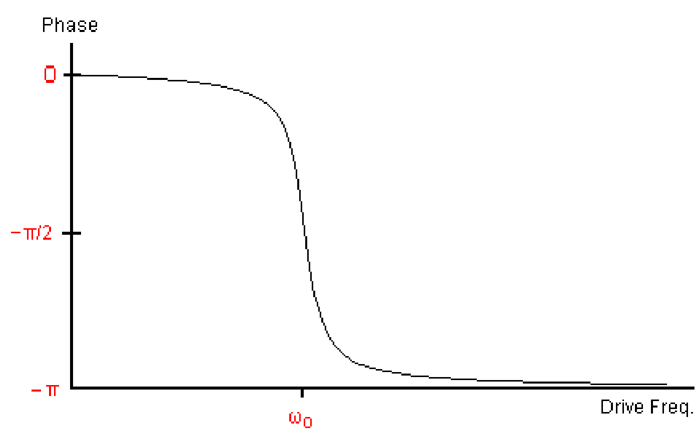
$$\phi(\infty) = \text{"}\tan^{-1}\frac{2\beta\cdot\infty}{\infty^2 - \omega_0^2}\text{"} = \text{"}\tan^{-1}\frac{2\beta}{\infty}\text{"} = \tan^{-1}(0) = -\pi.$$

Summary for Sketch:

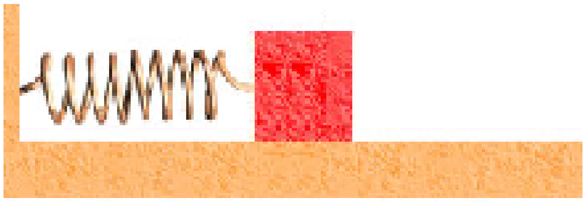
$$\phi(0) = 0$$

$$\phi(\omega_0) = -\frac{\pi}{2}$$

$$\phi(\infty) = -\pi$$



V2. Resonances.



Courtesy David M. Harrison
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The differential equation for the driven harmonic oscillator is identical in terms of its basic math when compared to the LRC circuit you just analyzed.

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=F_0\sin\omega t$$

$$L\frac{d^2q}{dt^2}+R\frac{dx}{dt}+\frac{q}{C}=V_0\sin\omega t$$

Compare to see the analogy

$$x\leftrightarrow q \qquad m\leftrightarrow L \qquad b\leftrightarrow R \qquad k\leftrightarrow\frac{1}{C} \qquad F_0\leftrightarrow V_0$$

$$q(t)=\frac{V_0}{L}\frac{\sin(\omega t+\phi)}{\sqrt{(\omega^2-\omega_0^2)^2+4\beta^2\omega^2}}$$

$$x(t)=\frac{F_0}{m}\frac{\sin(\omega t+\phi)}{\sqrt{(\omega^2-\omega_0^2)^2+4\beta^2\omega^2}}$$

$$\frac{d}{d\omega}\left[(\omega^2-\omega_0^2)^2+4\beta^2\omega^2\right]=0$$

$$2(\omega^2-\omega_0^2)2\omega+4\beta^2 2\omega=0$$

$$\left[(\omega^2-\omega_0^2)+2\beta^2\right]\omega=0$$

$$\omega^2 - \omega_0^2 + 2\beta^2 = 0$$

$$\omega^2 = \omega_0^2 - 2\beta^2 \qquad\qquad \omega_R^2 = \omega_0^2 - 2\beta^2$$

$$\omega_R = \sqrt{\omega_0^2 - 2\beta^2}$$

$$q(t) = C\frac{\omega_0^2}{\sqrt{(\omega^2-\omega_0^2)^2 + 4\beta^2\omega^2}} V_0 \sin(\omega t + \phi)$$

$$i(t) = C\frac{\omega_0^2}{\sqrt{(\omega^2-\omega_0^2)^2 + 4\beta^2\omega^2}} \omega V_0 \cos(\omega t + \phi)$$

Current is in Phase with Voltage so go for the Voltage across R. So we do the transfer function analysis for the output across R.

$$H(\omega) = \frac{V_{out}(t)}{V_{in}(t)} = \frac{R}{j\omega L + R + \frac{1}{j\omega C}}$$

$$H(\omega) = \frac{V_{out}(t)}{V_{in}(t)} = \frac{R}{j(\omega L - \frac{1}{\omega C}) + R}$$

$$\left|H(\omega)\right| = \frac{R}{(\omega L - \frac{1}{\omega C})^2 + R^2}$$

For max: $\omega L - \frac{1}{\omega C} = 0 \qquad \omega^2 = \frac{1}{LC} = \omega_0^2 \qquad \omega = \omega_0$