Theoretical Physics
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Chapter W Homework - Solutions. Principle of Least Action

W1. Abram Bader, Feynman's High School Physics Teacher.

"Bader asked Feynman to consider a less intuitive quantity than the sum of these energies: their difference. Subtracting the potential energy from the kinetic energy was as easy as adding them. It was just a matter of changing signs. But understanding the physical meaning was harder. Far from being conserved, this quantity the - *action* Bader said - changed constantly. Bader had Feynman calculate it for the ball's entire flight to the window. And he pointed out what seemed to Feynman a miracle. At any particular moment the action might rise or fall, but when the ball arrived at its destination, the path it had followed would always be the path for which the total action was least. For any other path Feynman might try drawing on the blackboard - a straight line from the ground to the window, a higher-arcing trajectory, or a trajectory that deviated however slightly from the fated path - he would find a greater average difference between kinetic and potential energy." ***James Gleick, The Life and Science of Richard Feynman (Vintage Books, New York. 1993)***

$$v = v_0 - gt \qquad x = v_0 t - \frac{1}{2} g t^2$$

$$L = \frac{1}{2} m v^2 - mgx$$

$$x(0) = 0 \qquad x(1) = 1 \qquad m = 1 \qquad v_0 = 2 \qquad g = 2$$

Given (a): $x = 2t - t^2 \qquad v = 2 - 2t \qquad L = \frac{1}{2} v^2 - 2x$

$$L = \frac{1}{2}(2-2t)^2 - 2(2t - t^2) = \frac{1}{2}(4 - 8t + 4t^2) - 4t + 2t^2$$

$$L = (2 - 4t + 2t^2) - 4t + 2t^2$$

$$L = 4t^2 - 8t + 2$$

$$S = \int_{t_1}^{t_2} L\,dt \qquad S = \int_0^1 L\,dt \qquad S = \int_0^1 (4t^2 - 8t + 2)dt$$

$$S_a = \left[\frac{4t^3}{3} - \frac{8t^2}{2} + 2t \right]_0^1 = \frac{4}{3} - 4 + 2 = \frac{4}{3} - 2 = \frac{4}{3} - \frac{6}{3} = -\frac{2}{3}$$

$$\boxed{S_a = -\frac{2}{3}}$$

Given (b): $x(t) = t$, then $v(t) = 1$.

$$L = \frac{1}{2}v^2 - 2x \qquad S = \int_0^1 (\frac{1}{2} - 2t)dt$$

$$S_b = \left[\frac{t}{2} - \frac{2t^2}{2} \right]_0^1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$\boxed{S_b = -\frac{1}{2}}$$

Given (c): $x(t) = t^2$, then $v(t) = 2t$.

$$L = \frac{1}{2}v^2 - 2x \qquad S_c = \int_0^1 (2t^2 - 2t^2)dt = 0$$

$$\boxed{S_c = 0}$$

Given (d): $x(t) = t^3$, then $v(t) = 3t^2$.

$$L = \frac{9}{2}t^4 - 2t^3 \qquad S = \int_0^1 (\frac{9}{2}t^4 - 2t^3)dt$$

$$S_d = \left[\frac{9t^5}{2 \cdot 5} - \frac{2t^4}{4} \right]_0^1 = \frac{9}{10} - \frac{1}{2} = \frac{9}{10} - \frac{5}{10} = \frac{4}{10} = \frac{2}{5}$$

$$\boxed{S_d = \frac{2}{5}}$$

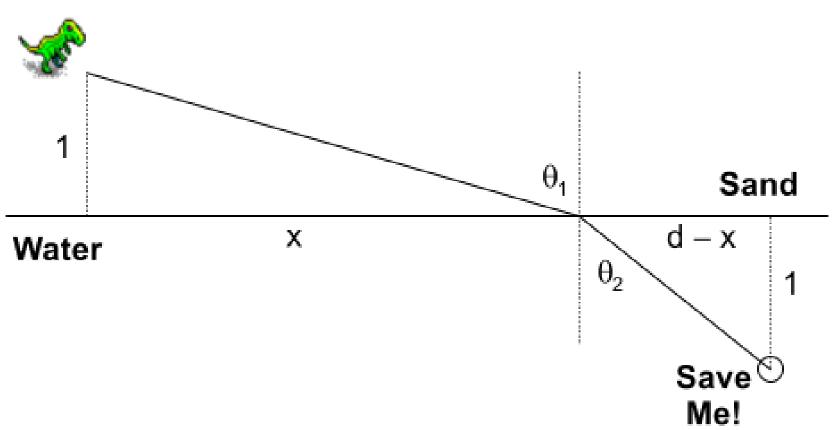
The actions from least to greatest are given from left to right.

S_a	S_b	S_c	S_d
$-\frac{2}{3}$	$-\frac{1}{2}$	0	$\frac{2}{5}$

The path realized in nature is the one with the least action, S_a .

$$S_a < S_b < S_c < S_d$$

W2. The Principle of Least Time.



$$t = t_1 + t_2 = \frac{d_1}{v_1} + \frac{d_2}{v_2} = \frac{\sqrt{x^2 + 1}}{v_1} + \frac{\sqrt{(d-x)^2 + 1}}{v_2}$$

Now use $n_1 = c / v_1$ and $n_2 = c / v_2$ as given definitions.

$$t = \frac{n_1}{c}\sqrt{x^2 + 1} + \frac{n_2}{c}\sqrt{(d-x)^2 + 1}$$

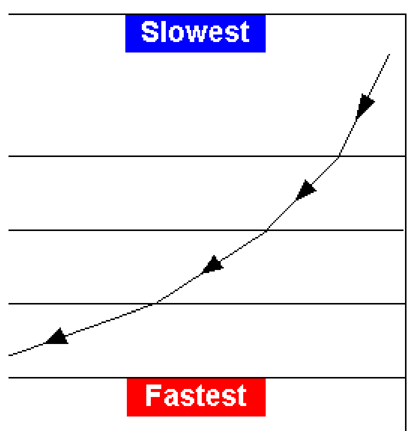
$$\frac{dt}{dx} = \frac{n_1}{c}\frac{1}{2}\frac{2x}{\sqrt{x^2+1}} + \frac{n_2}{c}\frac{1}{2}\frac{2(d-x)(-1)}{\sqrt{(d-x)^2+1}} = 0$$

$$\frac{dt}{dx} = \frac{n_1}{c}\frac{x}{\sqrt{x^2+1}} - \frac{n_2}{c}\frac{(d-x)}{\sqrt{(d-x)^2+1}} = 0$$

$$n_1 \frac{x}{\sqrt{x^2+1}} = n_2 \frac{(d-x)}{\sqrt{(d-x)^2+1}}$$

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Any sketch with layers is fine.



Our $t = t_1 + t_2 = \frac{d_1}{v_1} + \frac{d_2}{v_2} = \frac{n_1}{c} d_1 + \frac{n_2}{c} d_2 = \frac{n_1}{c} s_1 + \frac{n_2}{c} s_2$ becomes

$$t = \frac{1}{c} \sum n_i s_i$$

For the continuous case as our layers are infinitesimally small, we have

$$t = \frac{1}{c} \int n(s)\, ds \, .$$