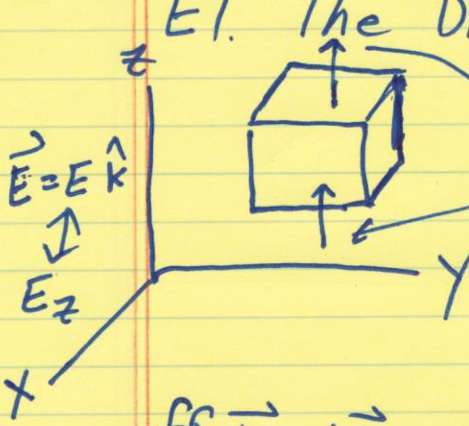
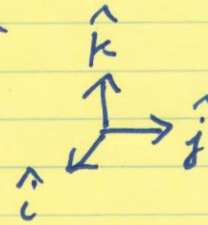
E. Differential Form for the Maxwell Equations

E1. The Divergence Theorem





Right-handed coordinate system) $\hat{i} \times \hat{j} = \hat{k}$

points outward from surface normal, i.e. ⊥

Top Panel $\Delta\vec{A} = \Delta x \Delta y \, \hat{k}$

Botton $\Delta\vec{A} = \Delta x \Delta y \, (-\hat{k})$

$$\oiint \vec{E} \cdot d\vec{A} = E_z(x, y, z+\Delta z)\, \Delta x \Delta y - E_z(x, y, z)\, \Delta x \Delta y$$

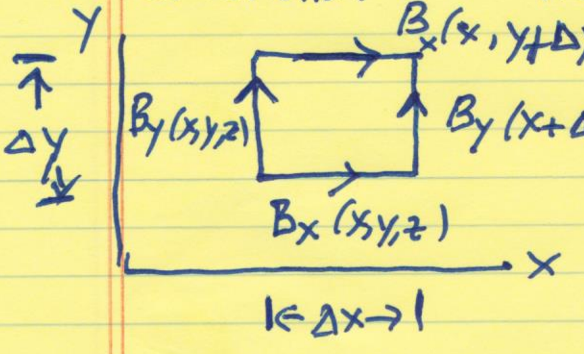
Multiply by $\frac{\Delta z}{\Delta z}$ $\Rightarrow$ $\frac{\Delta E_z}{\Delta z} \Delta z \Rightarrow \frac{\partial E_z}{\partial z} dx\,dy\,dz$

In general
$$\oiint \vec{E} \cdot d\vec{A} = \iiint_V \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) \underbrace{dx\,dy\,dz}_{dV}$$

$$\nabla \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\boxed{\oiint \vec{E} \cdot d\vec{A} = \iiint_V \nabla \cdot \vec{E}\, dV} \qquad dV = dx\,dy\,dz$$

E2. Stoke's Theorem



$\oint \vec{B} \cdot d\vec{\ell} \Rightarrow$ Counterclockwise integration

$$B_x(x,y,z)\Delta x + B_y(x+\Delta x, y, z)\Delta y$$
$$- B_x(x, y+\Delta y, z)\Delta x - B_y(x, y, z)\Delta y$$

$$[B_y(x+\Delta x, y, z) - B_y(x,y,z)]\Delta y - [B_x(x, y+\Delta y, z) - B_x(x,y,z)]\Delta x$$

Mult. by $\frac{\Delta x}{\Delta x}$ term 1 $\Rightarrow$ $\frac{\partial B_y}{\partial x} dx\,dy - \frac{\partial B_x}{\partial y} dx\,dy$

$+ \frac{\Delta y}{\Delta y}$ term 2

$$\oint \vec{B} \cdot d\vec{\ell} = \iint_A \underbrace{\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right)}_{(\nabla \times \vec{B})_z} dx\,dy$$

will show in E4

$$\oint \vec{B}\cdot d\vec{l} = \iint_A \overbrace{(\nabla\times\vec{B})_z}^{\nabla\times\vec{B}}\, \hat{k}\cdot \underbrace{\hat{k}\,dA}_{d\vec{A}}$$

$$\boxed{\oint \vec{B}\cdot d\vec{l} = \iint_A (\nabla\times\vec{B})\cdot d\vec{A}}$$

E3. Maxwell Equations

1. $\oiint \vec{E}\cdot d\vec{A} = Q/\varepsilon_0$

Charge density

$$\iiint \nabla\cdot\vec{E}\,dV = \iiint_V \frac{\rho}{\varepsilon_0}\,dV$$

$$\iiint \underbrace{\left(\nabla\cdot\vec{E} - \rho/\varepsilon_0\right)}_{0} dV = 0$$

$$\nabla\cdot\vec{E} = \rho/\varepsilon_0$$

arbitrary volume choices in general so integrand must vanish to make equation always true

2. $\oiint \vec{B}\cdot d\vec{A} = 0 \Rightarrow \nabla\cdot\vec{B} = 0$

3. $\underbrace{\oint \vec{B}\cdot d\vec{l}}_{\iint_A (\nabla\times\vec{B})\cdot d\vec{A}} = \mu_0 i + \mu_0\varepsilon_0 \frac{d\Phi_E}{dt}$

Current density J so $i = JA$

$i = \iint_A \vec{J}\cdot d\vec{A}$ $\quad \Phi_E \equiv \iint_A \vec{E}\cdot d\vec{A}$

$$\boxed{\nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0 \frac{\partial\vec{E}}{\partial t}}$$

4. $\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt}$ $\quad \boxed{\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$

by analogy to $\nabla\times\vec{B}$

Note $\Phi_B = \iint_A \vec{B}\cdot d\vec{A}$

E4. The del operator $\nabla \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

means definition

1. The Gradient

could be temperature

Considera scalar function $\phi = f(x, y, z)$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} \quad \text{a vector}$$

Change in temperature along the 3 dimensions

2. The Divergence

Consider a vector $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$A_x = A_x(x, y, z)$, $A_y = A_y(x, y, z)$, $A_z = A_z(x, y, z)$

$$\nabla\cdot\vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)$$

$\hat{i}\cdot\hat{i} = 1$ $\quad \hat{i}\cdot\hat{j} = \hat{i}\cdot\hat{k} = 0$

$$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad \text{is a scalar}$$

3. The Curl $\quad \nabla\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$

$$\nabla\times\vec{A} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{j}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

is a vector