Fourier Series — Periodic Waves

What about nonperiodic — A Slap? Nonperiodic function of time or space.

You need frequencies in between harmonics.

March 17, 2020 P-1

Class P. Fourier Transforms

P1. Fourier Series with Exponentials

Step. 1 Introducting Exponentials

From Last class

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right]$$

$e^{ix} = \cos x + i \sin x$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$\cos x = \frac{1}{2}(e^{x} + e^{-x}) \qquad \sin x = \frac{1}{2i}\left[e^{x} - e^{-x}\right]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty}\underbrace{\left[a_n \frac{1}{2}(e^{inx} + e^{-inx}) + b_n \frac{1}{2i}(e^{inx} - e^{-inx})\right]}$$

$$C_0 \qquad e^{inx}\underbrace{\tfrac{1}{2}(a_n - ib_n)}_{c_n \;\; n>0} + e^{-inx}\underbrace{\tfrac{1}{2}(a_n + ib_n)}_{c_n \;\; n<0}$$

$C_0 = \frac{a_0}{2}$

"Infant" will grow up to become an integral

a_n b_n equations

$$* \; f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx \quad n>0$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{inx}\,dx \quad n<0$$

Infant form "Fourier Transform" will grow up to be an integral from $-\infty$ to $+\infty$

$$* \; c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx \quad \text{for all } n$$

P2. Expanding the Interval

Step 2. Expanding the Interval

Change x to z first. $-\pi \le z \le \pi$, $-L \le x \le +L$

$$g(z) = \sum_{n=-\infty}^{\infty} c_n e^{inz} \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(z) e^{-inz}\,dz$$

$$\frac{z}{x} = \frac{\pi}{L} \qquad z = \frac{\pi}{L}x \qquad dz = \frac{\pi}{L}dz$$

$$g\left(\frac{\pi}{L}x\right) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{L}x} \qquad c_n = \frac{1}{2\pi}\int_{-L}^{L} g\left(\frac{\pi}{L}x\right) e^{-i\frac{n\pi}{L}x}\frac{\pi}{L}dx$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}} \qquad c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-\frac{in\pi x}{L}}\,dx$$

Fourier Transform can handle nonperiodic functions.

P3. Transforming to an Integral

Step 3. Transforming to an Integral

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \Delta n \qquad \Delta n = 1$$

$$f(x) = \int_{-\infty}^{\infty} C(n)\, e^{in\pi x/L}\, dn$$

i) $\Delta n \to dn$
ii) rip off n subscript
iii) turn Σ into a snake

promote c_n to function of continuous variable $\hookrightarrow C(n)$

$$k \equiv \frac{n\pi}{L} \quad n = \frac{Lk}{\pi} \quad dn = \frac{L}{\pi} dk$$

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} L\, C\!\left(\frac{L}{\pi}k\right) e^{ikx}\, dk$$

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} c(k)\, e^{ikx}\, dk$$

We assume in the next step that $Lc_n \to c(k)$ stays finite as $L \to \infty$. We will have to check for consistency

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-in\pi x/L}\, dx \quad \text{from before}$$

$$c(k) = \frac{1}{2}\int_{-L}^{L} f(x)\, e^{-ikx}\, dx$$

P4. Aiming for Infinity

Step 4. Aiming for Infinity. $L \to \infty$

$$c(k) = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

Must use different integration variable since x already appears.

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} c(k)\, e^{ikx}\, dk = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x')\, e^{-ikx'}\, e^{ikx}\, dx'\, dk$$

$$f(x) = \int_{-\infty}^{\infty} f(x') \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}\, dk}_{\delta(x-x')}\, dx' = f(x)$$

Consistency!

$\delta(x-x')$ ← Remember this one?

Share the $\frac{1}{2}$ + $\frac{1}{\pi}$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk \quad \leftarrow \text{Inverse Fourier Transform.}$$

Fourier Transform →

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

$$\mathcal{F}\{f(x)\} \equiv F(k) \qquad \mathcal{F}^{-1}\{F(k)\} = f(x)$$