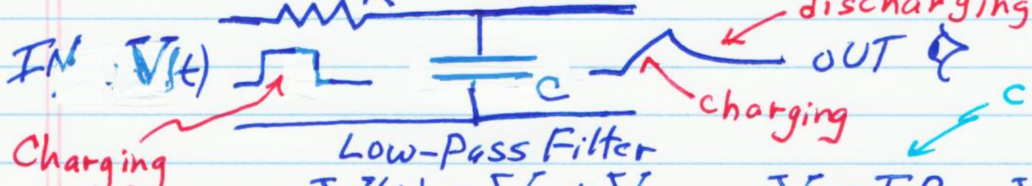
April 9, 2020

Class V. Transfer Functions

— Low Pass Filter

VI. Driven Oscillations and an LP Filter

$V(t)$ Voltage



Low-Pass Filter

$$V(t) = V_R + V_C \qquad V_R = IR \qquad V_C = \frac{q}{C} \qquad I = \frac{dq}{dt}$$

current — charge — Ohm's Law

Same Differential Equation as Radioactive Dumping

$$V(t) = IR + \frac{q}{C}$$

$$V(t) = R\frac{dq}{dt} + \frac{q}{C}$$

$$\frac{V(t)}{R} = \frac{dq}{dt} + \frac{q}{RC}$$

$$f(t) = \frac{dn(t)}{dt} + \lambda n(t)$$

Recall Radioactive dumping

$$\frac{dn(t)}{dt} = -\lambda n(t) + f(t)$$

We will consider a sinusoidal driving Voltage

$$f(t) = \frac{dq}{dt} + \lambda q \qquad \text{where } f(t) = \frac{V(t)}{R} \quad \lambda = \frac{1}{RC}$$

$$V(t) = V_0 \sin \omega t \qquad G(t,0) = e^{-\lambda t} \qquad f(t) = \frac{V_0 \sin \omega t}{R}$$

Solution

$$q(t) = \int_0^t f(t) e^{-\lambda(t-u)} du$$

$$\llcorner\ G(t,u) = e^{-\lambda(t-u)}$$

$$q(t) = \int_0^t \frac{V_0}{R} \sin(\omega u)\, e^{-\lambda(t-u)} du$$

$$\frac{R}{V_0} q(t) = \int_0^t \sin(\omega u)\, e^{-\lambda(t-u)} du$$

$$\to \cos(\omega u) + i\underbrace{\sin(\omega u)}_{\text{Imaginary Part}}$$

Take Imaginary Part

$$\frac{R}{V_0} q(t) = \mathrm{Im}\int_0^t e^{i\omega u} e^{-\lambda(t-u)} du$$

$$\frac{R}{V_0} q(t) = e^{-\lambda t}\, \mathrm{Im}\int_0^t e^{i\omega u} e^{+\lambda u} du$$

$$\frac{R}{V_0} q(t) = e^{-\lambda t}\, \mathrm{Im}\int_0^t e^{(\lambda + i\omega) u} du$$

$$\frac{R}{V_0} q(t) = e^{-\lambda t}\, \mathrm{Im} \left. \frac{e^{(\lambda + i\omega)u}}{\lambda + i\omega} \right|_0^t$$

$$\frac{R}{V_0}q(t) = e^{-\lambda t}\, Im\left[\frac{e^{(\lambda+i\omega)t} - 1}{\lambda+i\omega}\right]$$

Multiply top + bottom by Complex conjugate of $\lambda+i\omega$

$$\frac{R}{V_0}q(t) = e^{-\lambda t}\, \underbrace{Im\left[\frac{e^{(\lambda+i\omega)t} - 1}{\lambda+i\omega}\right]\frac{(\lambda-i\omega)}{(\lambda-i\omega)}}_{Im\left[\frac{\left(e^{\lambda t}e^{i\omega t}-1\right)(\lambda-i\omega)}{\lambda^2+\omega^2}\right]}$$

$e^{-\lambda t}e^{\lambda t}=1$

$$\frac{R}{V_0}q(t) = \underbrace{Im\left[1\frac{e^{i\omega t}}{\lambda^2+\omega^2}(\lambda-i\omega)\right]}_{\frac{1}{\lambda^2+\omega^2}\underbrace{Im\left[(\cos\omega t+i\sin\omega t)(\lambda-i\omega)\right]}_{-\omega\cos\omega t+\lambda\sin\omega t}} - e^{-\lambda t}\underbrace{Im\left(\frac{\lambda-i\omega}{\lambda^2+\omega^2}\right)}_{\frac{-\omega}{\lambda^2+\omega^2}}$$

Power of the convolution

$$\frac{R}{V_0}q(t) = \frac{\lambda\sin\omega t-\omega\cos\omega t}{\lambda^2+\omega^2} + \frac{\omega e^{-\lambda t}}{\lambda^2+\omega^2}$$

Check: Does $q(0)=0$ $\quad t=0 \Rightarrow \frac{0-\omega}{\lambda^2+\omega^2} + \frac{\omega}{\lambda^2+\omega^2} = 0$ ✓

$\lambda = \frac{1}{RC}$ $\quad RC \Rightarrow$ Time units

Recall $G(t,0) = e^{-\lambda t} = e^{-t/RC}$

Power of the Green's Function

$\omega_c \equiv \lambda$ $\quad \tau \equiv RC$ Time Characteristic of Circuit or cutoff timeframe

$\frac{1}{\tau} \Rightarrow$ frequency

↳ Characteristic frequency of the RC Circuit or cutoff frequency

↳ will see later

$$\frac{R}{V_0}q(t) = \underbrace{\frac{\omega_c\sin\omega t-\omega\cos\omega t}{\omega_c^2+\omega^2}}_{\text{Steady State Solution}} + \underbrace{\frac{\omega e^{-\omega_c t}}{\omega_c^2+\omega^2}}_{\text{Transient Solution}}$$

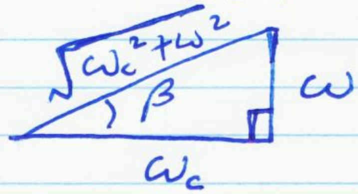
Complete Knowledge of what is going on.

As $t\to\infty$ $\quad \frac{\omega e^{-\omega_c t}}{\omega_c^2+\omega^2} \longrightarrow 0$

Transient Part of Solution eventually vanishes

Steady state part $\frac{R}{V_0} q(t) = \frac{\omega_c \sin\omega t - \omega \cos\omega t}{\omega_c^2 + \omega^2}$

Consider $\underbrace{\omega_c}_{\sim\cos\beta} \sin\omega t - \underbrace{\omega}_{\sim\sin\beta} \cos\omega t$ $\qquad \beta = \tan\frac{\omega}{\omega_c}$



$$\left.\begin{aligned}\omega_c &= \sqrt{\omega_c^2+\omega^2}\cos\beta \\ \omega &= \sqrt{\omega_c^2+\omega^2}\sin\beta\end{aligned}\right\}$$

Idea from $\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta$

Rotation Matrix Product

$$\begin{pmatrix}\cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\cos\beta & \sin\beta \\ -\sin\beta & \cos\beta\end{pmatrix} \Rightarrow \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Recall this trick. $\begin{bmatrix}\cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta)\end{bmatrix}$

$\sin(\alpha-\beta) = \cos\alpha\sin(-\beta) + \sin\alpha\cos(-\beta)$

$\sin(\alpha-\beta) = -\cos\alpha\sin\beta + \sin\alpha\cos\beta$

Compare $\omega_c \sin\omega t - \omega\cos\omega t$

to $\cos\beta\sin\alpha - \sin\beta\cos\alpha$

How do theorists come up with this step? A step like this?

$\alpha = \omega t \qquad \cos\beta = \frac{\omega_c}{\sqrt{\omega_c^2+\omega^2}} \qquad \sin\beta = \frac{\omega}{\sqrt{\omega_c^2+\omega^2}}$

Insight for the Trig Idea to simplify $\omega_c \sin\omega t - \omega\cos\omega t \sim q(t)$

1) Theoretical Intuition — Looks like $\sin(\alpha-\beta)$
2) Experimental Intuition — $q(t)$ should be a sinusoidal response being driven by a sinusoidal input but we expect $q(t)$ to be out of phase with the driving voltage $\Rightarrow \sin(\omega t - \beta)$ form

(ω: driving frequency; β: phase)

$$\frac{R}{V_0} q(t) = \frac{\omega_c \overbrace{}^{\sqrt{\omega_c^2+\omega^2}\cos\beta} \sin\omega t - \omega \overbrace{}^{\sqrt{\omega_c^2+\omega^2}\sin\beta} \cos\omega t}{\omega_c^2 + \omega^2}$$

$$\frac{R}{V_0} q(t) = \frac{\sin\omega t\cos\beta - \cos\omega t\sin\beta}{\sqrt{\omega_c^2+\omega^2}} = \frac{\sin(\omega t - \beta)}{\sqrt{\omega_c^2+\omega^2}}$$

$$q(t) = \frac{V_0}{R}\frac{\sin(\omega t-\beta)}{\sqrt{\omega_c^2+\omega^2}} \qquad V_c(t) = \frac{q(t)}{C}$$

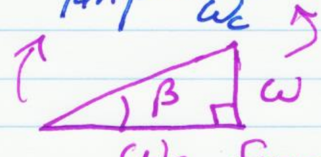
$$q(t) = \frac{V_0}{R}\frac{\sin(\omega t-\beta)}{\sqrt{\omega_c^2+\omega^2}} \qquad V_c(t) = \frac{q(t)}{C}$$

└ Voltage across the capacitor

$$V_c(t) = \frac{V_0}{RC}\frac{\sin(\omega t-\beta)}{\sqrt{\omega_c^2+\omega^2}} \qquad \frac{1}{RC} \equiv \omega_c$$

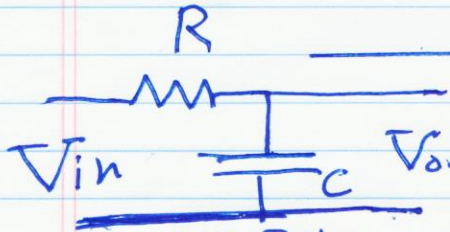
I like to write the variable ω first $\Rightarrow \sqrt{\omega^2+\omega_c^2}$

$$V_c(t) = \frac{V_0\,\omega_c \sin(\omega t-\beta)}{\sqrt{\omega^2+\omega_c^2}} \qquad \tan\beta = \frac{\omega}{\omega_c}$$

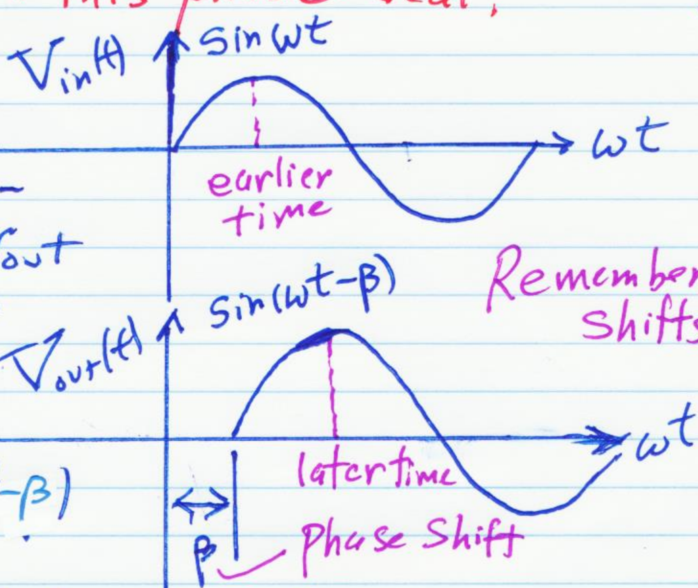


ω_c from before

What's with this phase deal?



LP Filter



Remember $f(x-d)$ shifts a function $f(x)$ to the right by d.

$V_{in} = V_0 \sin \omega t$

$V_{out} = V_c \sim \sin(\omega t - \beta)$

└ Lags behind by phase β

└ Engineers prefer a negative phase angle for lag

So they use $\phi = -\beta$

$$V_c(t) = V_0 \frac{\omega_c \sin(\omega t+\phi)}{\sqrt{\omega^2+\omega_c^2}} \qquad \tan\phi = \tan(-\beta) = -\frac{\omega}{\omega_c}$$

Transmission

$$T \equiv \left|\frac{V_{out}}{V_{in}}\right| = \left|\frac{\dfrac{V_0\,\omega_c \sin(\omega t+\phi)}{\sqrt{\omega^2+\omega_c^2}}}{V_0 \sin(\omega t)}\right| = \frac{\omega_c}{\sqrt{\omega^2+\omega_c^2}}$$

Magnitude of Ratio

Output compared to input

Low pass

$$\lim_{\omega\to 0}\left|\frac{V_{out}}{V_{in}}\right| = \lim_{\omega\to 0}\frac{\omega_c}{\sqrt{\omega^2+\omega_c^2}} = \frac{\omega_c}{\omega_c} = 1$$

└ Low frequencies

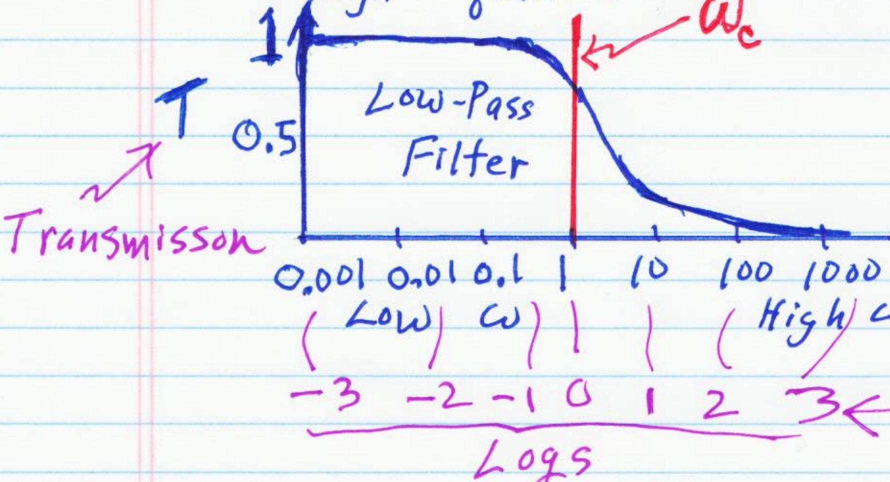
$$T = \left|\frac{V_{out}}{V_{in}}\right| = \frac{\omega_c}{\sqrt{\omega^2+\omega_c^2}}$$

$\omega_c \leftarrow$ also cutoff frequency

$$\lim_{\omega\to\infty}\left|\frac{V_{out}}{V_{in}}\right| = \lim_{\omega\to\infty}\frac{\omega_c}{\sqrt{\omega^2+\omega_c^2}} = 0$$

$\omega \to \infty$ ∟ high frequencies

high frequencies do not pass through



Note $\omega = 2\pi f$
∟ f frequency
∟ ω angular frequency

Note Horizontal Scale
Logarithmic Scale
Spacing – every step you multiply by 10

V2. Phasors

Engineers are clever and use phasors to get the above results without Green's function. We will look at this trick.
They are interested in $\left|\frac{V_{out}}{V_{in}}\right|$

$V = IR$

$V_0 \cos\omega t = i(t)R$

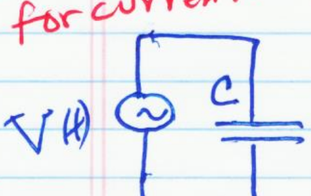
We will use
$V(t) = V_0\cos\omega t$ so when we move to $e^{i\omega t}$, the trig function that is real is what we have as driving force.

Now we honor electrical engineers and use $j = \sqrt{-1}$ Since $i(t)$ stands for current

Current is sacred "i" for current.

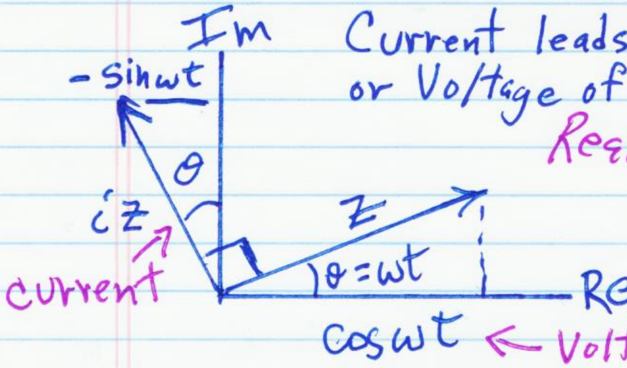
$$i(t) = \frac{V_0\cos\omega t}{R}$$

Current in sync with driving voltage



$$\frac{q(t)}{C} = V_0\cos\omega t \qquad i(t) = \frac{dq(t)}{dt} = -\omega C V_0 \sin\omega t$$

from $\frac{d}{dt}\cos\omega t$



Current leads the voltage by 90°
or Voltage of Source lags behind by 90°

– you have current first then charge build up $V_c \sim q$

Real Parts of z and iz give driving voltage and current

Set $V(t) = V_0 e^{j\omega t}$ phasor Rotation

$i(t) = j\omega C\, V(t)$

$\cos\omega t \leftarrow$ voltage $\quad V = Z_c\, i(t) \Longrightarrow Z_c = \frac{1}{j\omega C}$ Impedance

$V_{in}(t) \rightarrow$  $\rightarrow$ out & $V_{out} = V_c = q/C$

V3. Transfer Function with Phasors

Phasors $V_{in}(t) = i(t)\, Z_R + i(t)\, Z_c$ $Z_R \rightarrow R$ $Z_c \rightarrow \frac{1}{j\omega C}$

Generalized Ohm's Law with Impedance

$$\left.\begin{aligned} V_{in}(t) &= i(t)\left[R + \frac{1}{j\omega C}\right] \\ V_{out} &= i(t)\left[\frac{1}{j\omega C}\right] \end{aligned}\right\} \quad H(\omega) = \frac{V_{out}(t)}{V_{in}(t)}$$

← We think $V_c = i(t) Z_c$ instead of $q(t)/C$

$$H(\omega) = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{j\omega RC + 1} = \frac{1}{j\,\omega/\omega_c + 1} \qquad \omega_c = \frac{1}{RC} \quad RC = 1/\omega_c$$

$$H(\omega) = \frac{\omega_c}{j\omega + \omega_c}\left[\frac{-j\omega + \omega_c}{-j\omega + \omega_c}\right] = \omega_c\left(\frac{\omega_c - j\omega}{\omega^2 + \omega_c^2}\right)$$

Transmission

$$T = |H(\omega)| = \frac{\omega_c\sqrt{\omega_c^2 + \omega^2}}{\omega^2 + \omega_c^2} = \frac{\omega_c}{\sqrt{\omega^2 + \omega_c^2}}$$

What we got before

$$\text{Phase} \quad \tan\phi = \frac{\operatorname{Im} H(\omega)}{\operatorname{Re} H(\omega)} = -\frac{\omega}{\omega_c}$$

What we got before.

V4. Transfer Functions + Transforms

Here is a third way to get the transmission. This method gives insight into the Fourier transform.

Feynman — importance for theoretical physicist to know several ways to derive the same result!

Return to Good old physics.

$$v_{in}(t) = \frac{dq}{dt}R + \frac{q}{C} \qquad \text{Kirchhoff's Loop Rule}$$

$$v_{out}(t) = q/C$$

Take Fourier Transform

$$\mathcal{F}\{v_{in}(t)\} = \mathcal{F}\left\{\frac{dq}{dt}R\right\} + \mathcal{F}\left\{\frac{q}{C}\right\}$$

$$\mathcal{F}\{v_{out}(t)\} = \mathcal{F}\{q/C\} \Rightarrow V_{out}(\omega) = \frac{1}{C}Q(\omega)$$

$j \equiv \sqrt{-1}$

$$V_{in}(\omega) = j\omega R Q(\omega) + Q(\omega)/C$$

$$H(\omega) \equiv \frac{V_{out}(\omega)}{V_{in}(\omega)} = \frac{Q(\omega)/C}{j\omega R Q(\omega) + Q(\omega)/C} = \frac{1}{j\omega RC + 1}$$

Same result as before

Same

Transform Space

$$H(\omega) = \frac{1}{j\frac{\omega}{\omega_c} + 1} = \frac{\omega_c}{\omega_c + j\omega}$$

This method is very elegant + short!