Class W. The Principle of Least Action

W1. Gravity, Time, and Lagrangians.

Reference: Elisha Huggins (2010) a student of Feynman → The Physics Teacher **48**, pp 512-514 (November 2010)

First, a Review

Work - pick up a stone "+" ↑ $W = Fd = mgh$ (force, distance)

Now drop the stone.

$F = ma = m\frac{dv}{dt}$

$$W = \int_z^0 \underbrace{-m\frac{dv}{dt}}_{\text{force is down}} dz = \int_0^z m\frac{dv}{dt}dz = \int_0^t m\frac{dv}{dt}\frac{dz}{dt}dt$$

Think: $\frac{\Delta v}{\Delta t}\underbrace{\frac{\Delta z}{\Delta t}}_{v}\Delta t$ → $v\frac{\Delta v}{\Delta t}\Delta t$ → $v\Delta v$

$$W = \int_0^v mv\,dv = \frac{mv^2}{2}\Big|_0^v = \frac{1}{2}mv^2$$

You can choose "+" as down.

$W = \int_0^z m\frac{dv}{dt}dz$

$W = \frac{1}{2}mv^2$ same answer

Kinetic energy $\frac{1}{2}mv^2$

Potential energy mgh

Total Energy $0 + \underbrace{mgh}_{\text{initial height}} = \frac{1}{2}mv^2 + 0$

Drop from rest. Usual $h=0$ is ground.

Ground $h=0$

right before you crash into the ground

General Form: $\frac{1}{2}mv_1^2 + mgh_1 = \frac{1}{2}mv_2^2 + mgh_2$

Let photon go down from 1 to 2 (ph.)

Red photon will not speed up but become bluer since $E = hf$ for photons as potential energy is converted into photon energy. The frequency f increases.

$$\underbrace{hf_1}_{\text{photon energy}} + mgz = hf_2 + 0$$

But $E = mc^2$

hf ‖ effective mass of photon

$\Rightarrow m = hf/c^2$

$$hf_1 + \frac{hf_1}{c^2}gz = hf_2$$

h cancels

$$f_2 = f_1\left(1 + \frac{gz}{c^2}\right)$$

$f = \frac{1}{T}$ $T = \frac{1}{f}$ Period is T

$$T_1 = T_2\left(1 + \frac{gz}{c^2}\right)$$

Gain in time

$$\Delta T_{GR} = \frac{gz}{c^2}$$

GR: General Relativity

Clock at height z gains time.

Recall special relativity $T_{Lab} = \dfrac{T_0}{\sqrt{1 - v^2/c^2}}$

└ Time dilates, stretch in lab
we age more in lab

$$T_0 = \sqrt{1 - v^2/c^2}\; T_{Lab}$$

↳ Traveling one ages less

$$(1-\epsilon)^{1/2} = 1 - \frac{1}{2}\epsilon \quad \text{small}$$

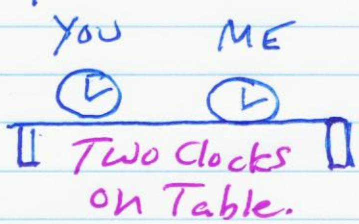
$$T_{moving} = \left(1 - \frac{1}{2}\frac{v^2}{c^2}\right) T_{Lab}$$ using Taylor Expansion

└ $\Delta T_{SR} = -\dfrac{v^2}{2c^2}$ You lose time moving, age less

Summary $\Delta T_{SR} = -\dfrac{v^2}{2c^2}$ $\Delta T_{GR} = \dfrac{gz}{c^2}$

└ lose time moving └ gain time at higher height

Feynman Game



You take a clock and I take one. We each travel with our clocks going wherever we want but we must bring our clocks back in one hour according to the roomclock.
The winner is the one whose clock gains the most time.

Strategy
1. You should move your clock up as high as you can.
2. It is a waste to move sideways. It loses time.
3. Don't speed up too much vertically. That loses time.

$$\Delta T_{player} = \underbrace{\Delta T_{GR}}_{\text{gain}} + \underbrace{\Delta T_{SR}}_{\text{Loss}}$$

$$\Delta T_{GR} = \frac{gz}{c^2} \qquad \Delta T_{SR} = -\frac{v^2}{2c^2}$$

3600 seconds (1 hour)

$$\Delta T_{score} = \sum_{n=1}^{3600}\left[\frac{g z_n}{c^2} - \frac{v_n^2}{2c^2}\right]$$

$$\Delta T_{Score} = \sum_{n=1}^{3600}\left[\frac{g z_n}{c^2} - \frac{v_n^2}{2c^2}\right]\Delta n \qquad \Delta n = 1$$

$$\Delta T_{score} = \sum_{n=1}^{3600} \left[\frac{g z_n}{c^2} - \frac{v_n^2}{2c^2} \right] \Delta n$$

$$\Delta T_{score} = \int_0^{1h} \left(\frac{g z(n)}{c^2} - \frac{v^2(n)}{2c^2} \right) dn$$

$dn \to dt$

Change notation $n \to t$

$$\Delta T_{score} = \int_0^{1h} \left(\frac{g z(t)}{c^2} - \frac{v^2(t)}{2c^2} \right) dt$$

Want to maximize

Multiply by $-mc^2$

$$\underbrace{-mc^2 \Delta T_{score}}_{S} = \int_0^{1h} \left[\frac{1}{2} m v^2(t) - mg z(t) \right] dt$$

Want to minimize due to the $-mc^2$

negative changes max to min

S is the action

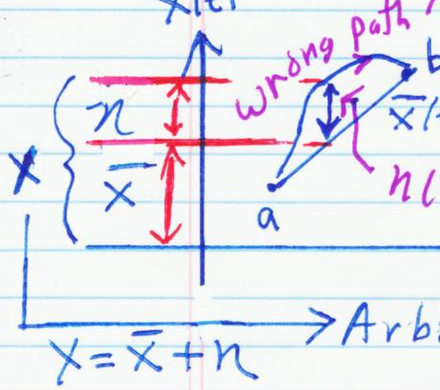
$$S = \int_a^b \underbrace{\left[\frac{1}{2} m v^2(t) - V(x) \right]}_{\text{Lagrangian}} dt$$

general form potential energy

$$\boxed{L = \frac{1}{2} m v^2(t) - V(x)}$$

W2. Least Action

Reference: The Feynman Lectures Vol 2, Ch 19
presented at Caltech 1961-1963



$\bar{x}(t)$ Ideal Path - the minimum case $\bar{x}(t)$

$\eta(x)$ Nonideal path deviation $\eta(t)$

Arbitrary Path $X(t) = \underbrace{\bar{x}(t)}_{\text{correct answer}} + \underbrace{\eta(t)}_{\text{deviation from correct answer}}$

Note: $X(t_a) = \bar{X}(t_a) \Rightarrow \eta(t_a) = 0$
$X(t_b) = \bar{X}(t_b) \Rightarrow \eta(t_b) = 0$

You must start at a and end up at b for all the paths under investigation

$$\frac{d\,X(t)}{dt} = \frac{d\,\bar{x}(t)}{dt} + \frac{d\eta(x)}{dt}$$

$$V(x) = V(\bar{x} + \eta)$$

$\eta(t_a) \equiv \eta_a = 0$
$\eta(t_b) \equiv \eta_b = 0$

$$S = \int_a^b \left[\frac{1}{2} m\left(\frac{dx}{dt}\right)^2 - V(x)\right] dt \qquad x = \bar{x} + \eta$$

General path

$$S = \int_a^b \left[\frac{1}{2} m\left(\frac{d\bar{x}}{dt} + \frac{d\eta}{dt}\right)^2 - \underbrace{V(\bar{x}+\eta)}_{V(\bar{x}) + V'(\bar{x})\eta + \frac{1}{2}V''(\bar{x})\eta^2 + \dots}\right] dt$$

Small

Taylor Series Expansion

$$S = \int_a^b \left[\frac{1}{2} m\left(\frac{d\bar{x}}{dt}\right)^2 + m\frac{d\bar{x}}{dt}\frac{d\eta}{dt} - V(\bar{x}) - V'(\bar{x})\eta\right] dt$$

We threw away $\frac{1}{2} m\left(\frac{d\eta}{dt}\right)^2$ and higher powers in the $V(x)$ expansion as they are extra small.

The delta or deviation

$$\delta S = \int_a^b \left[m\frac{d\bar{x}}{dt}\frac{d\eta}{dt} - V'(\bar{x})\,\eta\right] dt$$

arbitrary

need to free up the arbitrary η

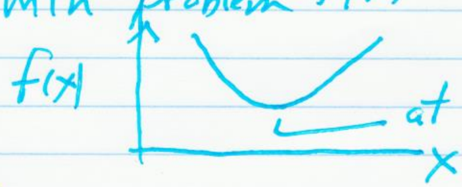
arbitrary paths

Note: We are justified in tossing η^2 and higher terms

" " " $\left(\frac{d\eta}{dt}\right)^2$ and " "

Since when $\delta S = 0$, deviations near the ideal will be small, i.e., small η + $\frac{d\eta}{dt}$

Think min problem $f(x)$



at min $\delta f(x) = 0 \qquad \frac{df}{dx} = 0$

To free up $\frac{d\eta}{dt}$: $\underbrace{\frac{d}{dt}\left[\frac{d\bar{x}}{dt}\eta\right]}_{\text{Product rule}} = \frac{d^2\bar{x}}{dt^2}\eta + \underbrace{\frac{d\bar{x}}{dt}\frac{d\eta}{dt}}_{\text{We have this one in our integral}}$

(Integration by parts.

$$\delta S = \int_a^b \left[\underbrace{m\frac{d}{dt}\left(\frac{d\bar{x}}{dt}\eta\right)}_{m\frac{d\bar{x}}{dt}\eta\Big|_a^b = 0} - m\frac{d^2\bar{x}}{dt^2}\eta - V'(\bar{x})\eta\right] dt = 0$$

since $\eta_a = \eta_b = 0$

$$\delta S = \int_a^b \underbrace{\left[-m\frac{d^2\bar{x}}{dt^2} - V'(\bar{x})\right]}_{\text{zero}} \eta\, dt = 0$$

arbitrary paths

For the path minimizing the action

$$-m\frac{d^2x}{dt^2} - V'(x) = 0$$

$$m\frac{d^2x}{dt} = -\frac{dV}{dx}$$ Conservative Force

$\underbrace{m\frac{d^2x}{dt}}_{a}$ acceleration; $\frac{dV}{dx} \to$ Force $F = -\frac{dV}{dx}$

$$ma = F$$

We get $F = ma$ Newton's 2nd Law

W3. The Lagrangian

Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - V(x) \qquad \dot{x} \equiv \frac{dx}{dt}$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{\partial L}{\partial x} = -\frac{dV}{dx}$$

momentum Force

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} = m\frac{d^2x}{dt^2} = ma$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}}$$ Euler-Lagrange Equation

In 3D, there will be one for each dimension.
The Euler-Lagrange Equations

Advanced form for $ma = F$ or $F = ma$
and very Powerful
Since you avoid vector force diagrams that can get nasty.