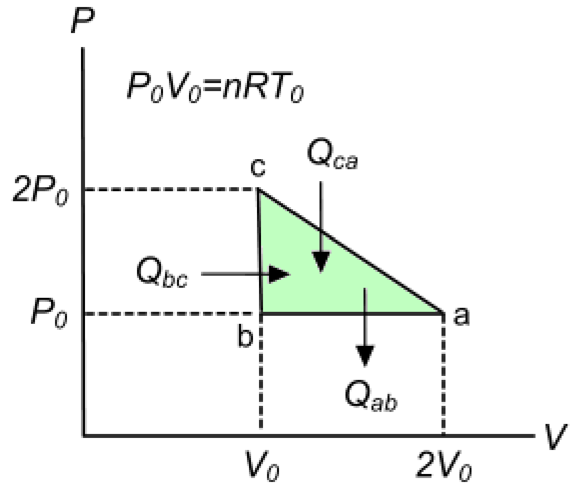
Theoretical Physics **Exam 2** **October 18, 2011**
Prof. Ruiz, UNC Asheville **Closed Book - No Calculator**

1. Engine. The total work W done by the engine during one "a-b-c" cycle is given by the shaded green region. The gas is ideal: PV = nRT and U = 3nRT/2.

a. Express W in terms of nRT_0.

b. Give a short sentence explaining why heat must flow into the system for the path b-c.

c. Calculate the total heat flowing in from b to c.



2. Statistical Mechanics. There are N = 26,000,000 particles, where some are in energy state $E_1 = 0$ and the rest are in state $E_2 = \varepsilon$. Give the occupation number n_1 for the 1st state when $\varepsilon = kT$, using the approximation $e = 2.718... \approx 2\frac{5}{7} = \frac{19}{7}$.

3. Commutator. In class we showed that $[x, p]\psi = i\hbar\psi$, i.e., $[x, p] = i\hbar$.

Express the commutator $\left[x^2, x\frac{d}{dx}\right]$ in simplest terms.

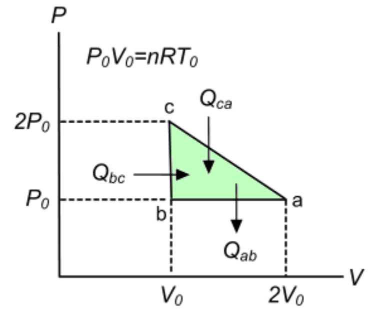
4. Dirac Four-Spinor. When a spin-1/2 is moving, the Dirac four spinor for the particle is given by

$$u(\vec{p}, s) = \sqrt{\frac{E+mc^2}{2E}}\begin{bmatrix} \phi(s) \\ \dfrac{c\vec{\sigma}\cdot\vec{p}}{E+mc^2}\phi(s) \end{bmatrix}, \text{ where } \phi(s) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Express this four-spinor in simplest terms for the case where

$$E = 2,\ mc^2 = 1,\ c\vec{p} = \hat{i} + \hat{j} + \hat{k}, \text{ and } \phi(s) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Theoretical Physics, Ruiz **Exam 2 - Solutions** **October 18, 2011**



1. Engine. The work done by the engine (green) is the green area: $\frac{1}{2}(2P_0 - P_0)(2V_0 - V_0) = \frac{1}{2}P_0V_0 = \frac{1}{2}nRT_0$

Heat must flow in for the path b-c since the temperature increases and thus the energy of the gas increases. This heat is $\Delta Q = \Delta U + \Delta W = \Delta U = \frac{3}{2}nR\Delta T = \frac{3}{2}nRT_0$.

2. Statistical Mechanics. $E_1 = 0$, $E_2 = \varepsilon$, $\varepsilon = kT$, $N = 26{,}000{,}000$, $e \approx 19/7$.

$$Z = \sum_i e^{-\frac{E_i}{kT}} = 1 + e^{-\frac{\varepsilon}{kT}} = 1 + e^{-1} = 1 + \frac{7}{19} = \frac{26}{19}$$

$$n_1 = Ne^{-\frac{E_1}{kT}} / Z = N / Z = 26{,}000{,}000\left[\frac{19}{26}\right] = 19{,}000{,}000$$

3. Commutator.

$$\left[x^2, x\frac{d}{dx}\right]\psi = x^2(x\frac{d\psi}{dx}) - x\frac{d}{dx}(x^2\psi) = x^3\frac{d\psi}{dx} - x(2x\psi) - x^3\frac{d\psi}{dx}$$

$\left[x^2, x\frac{d}{dx}\right]\psi = -x(2x\psi)$. Therefore, $\left[x^2, x\frac{d}{dx}\right] = -2x^2$.

4. Dirac Four-Spinor. $E = 2$, $mc^2 = 1$, $c\vec{p} = \hat{i} + \hat{j} + \hat{k}$, and $\phi(s) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$u(\vec{p}, s) = \sqrt{\frac{E + mc^2}{2E}}\begin{bmatrix} \phi(s) \\ \frac{c\vec{\sigma}\cdot\vec{p}}{E + mc^2}\phi(s) \end{bmatrix} \qquad \sqrt{\frac{E + mc^2}{2E}} = \sqrt{\frac{2+1}{2(2)}} = \frac{\sqrt{3}}{2}$$

$$c\vec{\sigma}\cdot\vec{p} = \sigma_x + \sigma_y + \sigma_z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

$$\frac{c\vec{\sigma}\cdot\vec{p}}{E + mc^2}\phi(s) = \frac{1}{3}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$ and $$u(\vec{p}, s) = \frac{\sqrt{3}}{2}\begin{bmatrix} 1 \\ 0 \\ 1/3 \\ (1+i)/3 \end{bmatrix}$$