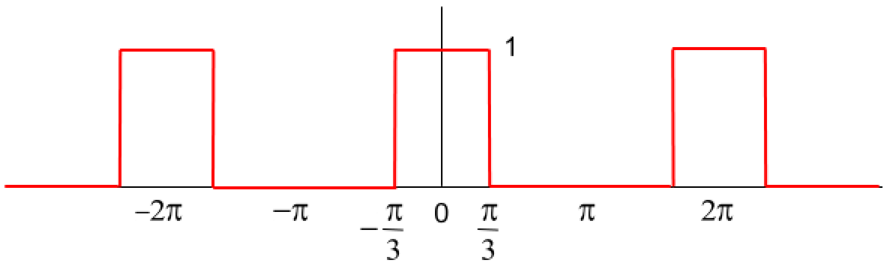
Theoretical Physics **Exam 3** **November 22, 2011**
Prof. Ruiz, UNC Asheville **Closed Book - No Calculator**

1. Fourier Series. Find the Fourier Series for the periodic wave shown below. Your basic cycle for this repeating pattern is defined over our standard region $-\pi \le x \le \pi$, symmetric about the origin, where the pulse is 1/3 the period (or wavelength) of the periodic wave.



Pulse Train with Duty Cycle $33\frac{1}{3}\%$

For full credit, write out your answer by giving the first six nonzero terms. You must write f(x) = and then give the coefficients multiplied by the appropriate trig function for 6 nonzero terms where each coefficient is in simplest mathematical form.

2. Convolution. Sketch the square pulse where $f(t) = 1$ for $0 \le t \le 1$ and zero elsewhere. Sketch the ramp pulse where $g(t) = t$ for $0 \le t \le 1$ and zero elsewhere. Find the convolution $h(t) = f * g$ inside the nonzero region $0 \le t \le 1$. Note that you can do $h(t) = g * f$ if that is easier. To gain insight into the convolution, imagine the square pulse moving in from the far left and overlapping your ramp as the square pulse travels to the right. Sketch an overlap where the right edge of the square pulse is at some arbitrary t in the region $0 \le t \le 1$. Your left edge will be at $t - 1$. What is the area of the common overlapped region?

3. Green's Function. Find the Green's function for the following differential equation in simplest form.

$$-\frac{d^2x}{dt^2} - \frac{dx}{dt} + 2x = f(t)$$

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1. Fourier Series. Square Pulse Train with $33\frac{1}{3}\%$ duty cycle with pulse center on zero.

This means an even function so that we only have the constant and the cosines.

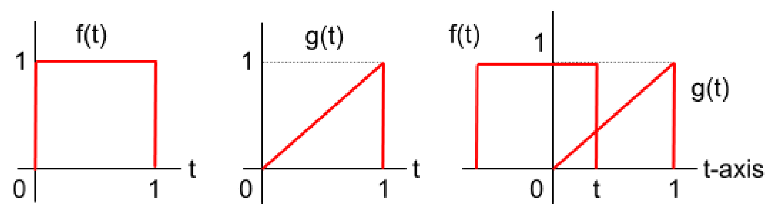
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi/3} 1\cdot dx = \frac{2}{\pi}x\Big|_0^{\pi/3} = \frac{2}{\pi}\frac{\pi}{3} = \frac{2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi/3}\cos(nx)\,dx = \frac{2}{\pi}\frac{\sin(nx)}{n}\Bigg|_0^{\pi/3} = \frac{2}{\pi}\frac{\sin(n\pi/3)}{n}$$

For n = 1, 2, 3, 4, 5, 6, 7, ... $\sin(n\pi/3)$ gives $\frac{\sqrt{3}}{2}$, $\frac{\sqrt{3}}{2}$, 0, $-\frac{\sqrt{3}}{2}$, $-\frac{\sqrt{3}}{2}$, 0, $\frac{\sqrt{3}}{2}$, ...

$$f(x) = \frac{1}{3} + \frac{\sqrt{3}}{\pi}\left[\cos x + \frac{\cos 2x}{2} - \frac{\cos 4x}{4} - \frac{\cos 5x}{5} + \frac{\cos 7x}{7} + \ldots\right]$$

2. Convolution.



$$f(t) * g(t) = \int_0^t f(u)g(t-u)du$$

$$f(t) * g(t) = \int_0^t 1\cdot(t-u)du$$

$$f(t) * g(t) = (tu - \frac{u^2}{2})\Big|_0^t$$

$$f(t) * g(t) = t^2 - \frac{t^2}{2} = \frac{t^2}{2}$$

The overlap area above is $\frac{1}{2}(t)(t) = \frac{t^2}{2} = f * g$.

3. Green's Function. $-\frac{d^2x}{dt^2} - \frac{dx}{dt} + 2x = \delta(t)$

Fourier Transform: $-(i\omega)^2 X(\omega) - i\omega X(\omega) + 2X(\omega) = \frac{1}{\sqrt{2\pi}}$

$$X(\omega) = \frac{1}{\sqrt{2\pi}}\frac{1}{\omega^2 - i\omega + 2} = \frac{1}{\sqrt{2\pi}}\frac{1}{(\omega + i)(\omega - 2i)} \qquad G(t) = \Im^{-1}\{X(\omega)\}$$

$$G(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\frac{e^{i\omega t}}{(\omega + i)(\omega - 2i)}d\omega \qquad G(t) = \frac{1}{2\pi}\oint\frac{e^{izt}}{(z+i)(z-2i)}dz$$

$$G(t) = \frac{1}{2\pi}2\pi i Res\left[\frac{e^{izt}}{(z+i)(z-2i)}, 2i\right] = \frac{1}{2\pi}2\pi i\frac{e^{izt}}{(z+i)}\Bigg|_{z=2i} = \frac{e^{-2t}}{3}$$