Theoretical Physics **Exam 1** **February 18, 2020**
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1. Matrices. Calculate the commutator $\left[\sigma_x, \sigma_y\right] \equiv \sigma_x \sigma_y - \sigma_y \sigma_x$, where

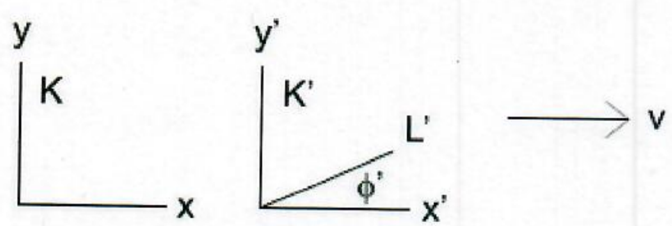
$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Express your final answer in terms of σ_x , σ_y , and σ_z .

2. Integral. Use a derivative trick with

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^2 + a^2} dx = \frac{\pi}{a} e^{-ma}$$ where $a > 0$, to evaluate $$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 1} dx .$$

3. Relativity. When the length of a rod along the x-axis in a moving frame K' is measured from the K frame, there is a Lorentz contraction of the form $L_x = L_x' \sqrt{1 - \frac{v^2}{c^2}}$. For a rod slanted in the K' frame at angle $\phi' = 30°$, what is the measured angle ϕ from the K frame for the velocity given below?



$$v = \frac{2\sqrt{2}}{3} c$$

4. Vector Calculus.

The vector $\vec{A} = x z \hat{i}$ and $\nabla \times \vec{A} = \frac{\partial \vec{B}}{\partial x}$, where the variables x, y, and z are all independent of each other. Find $\vec{B}$.

Theoretical Physics, Ruiz **Exam 1 - Solutions** **February 18, 2020**

1. Matrices. $\left[\sigma_x, \sigma_y\right] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\left[\sigma_x, \sigma_y\right] = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} = 2i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 2i\sigma_z \,.$$

2. Integral.

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+1}\,dx = \lim_{m\to 1}\left[-\frac{d}{dm}\right]\left[\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+1}\,dx\right] = \lim_{m\to 1}\left[-\frac{d}{dm}\right]\left[\pi e^{-m}\right]$$

$$= \lim_{m\to 1}\left[\pi e^{-m}\right] = \pi e^{-1} = \pi / e \qquad \text{or}$$

$$\lim_{\substack{m\to 1\\ a\to 1}}\left[-\frac{d}{dm}\right]\left[\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+a^2}\,dx\right] = \lim_{\substack{m\to 1\\ a\to 1}}\left[-\frac{d}{dm}\right]\left[\frac{\pi}{a}e^{-ma}\right] = \lim_{\substack{m\to 1\\ a\to 1}}\left[\frac{\pi}{a}ae^{-ma}\right] = \frac{\pi}{e}$$

3. Relativity.

$$L_x = L_x'\sqrt{1-\frac{v^2}{c^2}}\,,\; L_y = L_y'\,,\; \tan\phi' = L_y'/L_x'$$

$$\tan\phi = \frac{L_y}{L_x} = \frac{L_y'}{L_x'\sqrt{1-\frac{v^2}{c^2}}} = \frac{\tan\phi'}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{For } \phi' = 30^\circ \text{ and } v = \frac{2\sqrt{2}}{3}c$$

$$\tan\phi = \frac{(1/2)/(\sqrt{3}/2)}{\sqrt{1-\left[\frac{2\sqrt{2}}{3}\right]^2}} = \frac{1/\sqrt{3}}{\sqrt{1-\frac{8}{9}}} = \frac{1/\sqrt{3}}{\sqrt{\frac{1}{9}}} = \frac{3}{\sqrt{3}} = \sqrt{3} \text{ so } \phi = 60^\circ$$

4. Vector Calculus. $\vec{A} = x\,z\,\hat{i}$ and. $\nabla\times\vec{A} = \partial\vec{B}/\partial x$

$$\nabla\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & 0 & 0 \end{vmatrix} = x\,\hat{j} = \partial\vec{B}/\partial x$$

$$B_y = \int x\,dx = \frac{x^2}{2} + const$$

$$\vec{B} = \left[\frac{x^2}{2} + B_0\right]\hat{j} + f(y,z)\hat{i} + g(y,z)\hat{k}$$