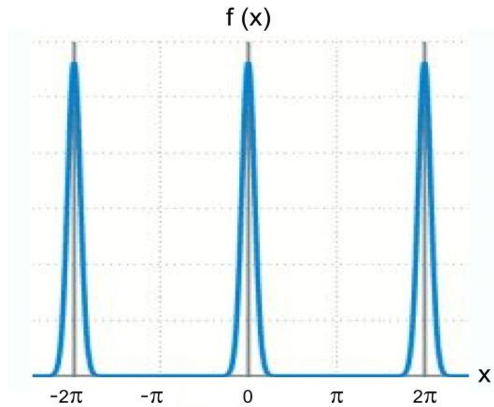
Theoretical Physics **Exam 3** **April 14, 2020**
Prof. Ruiz, UNC Asheville **Closed Book - No Calculator**

[15] Q1. Fourier Series. Find the Fourier Series for the periodic wave $f(x)$ shown at the right, where you take the peaks to be delta functions. Write out the first 5 terms of the Fourier series in their simplest forms.



[10] Q2. Fourier Transform. Now remove all the delta functions in the figure of Q1 except for the one at 2π. Find the Fourier transform $F(k)$ for your delta function at 2π. Give the result for $F(1)$, i.e., when $k=1$.

[15] Q3. Laplace Transform. Use the derivative trick with integrals to show that the Laplace transform of t is $\frac{1}{s^2}$, i.e., $L\{t\}=F(s)=\frac{1}{s^2}$. You can use this result in Q4 below.

[15] Q4. Laplace Transform of a Differential Equation. A car starts from rest at $x(0)=x_0$ and accelerates with constant acceleration $a>0$. The differential equation that describes this scenario is $\frac{dx}{dt}=at$. Take the Laplace transform of the differential equation and obtain the Laplace transform of $x(t)$, which you can call $X(s)$. Now find $x(t)$ from the Laplace Table, where you find $L\{t^n\}=\frac{n!}{s^{n+1}}$. Use this table information to obtain the solution $x(t)$.

[15] Q5. Convolution. Show that the "derivative trick" can be used to express the convolution of a radioactive-waste dumping function $f(t)=t$ and impulse radioactive-decay response $g(t)=e^{-\lambda t}$ as $f(t)*g(t)=e^{-\lambda t}\frac{d}{d\lambda}\left[\frac{e^{\lambda t}-1}{\lambda}\right]$.

[10] Q6. Cauchy-Riemann. Show that $z^2=(x+iy)^2$ is analytic.

[20] Q7. Complex Integration. Using complex integration, evaluate the definite integral $I=\int_{-\infty}^{\infty}\frac{dx}{x^4+4}$. If you have trouble, you may for half credit evaluate $I=\int_{-\infty}^{\infty}\frac{dx}{x^2+4}$ instead.

Theoretical Physics - Ruiz **Exam 3 - Solutions** **April 14, 2020**

1. Fourier Series. Dirac delta impulse train. Periodic $f(x)=\delta(x)$. $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos(nx)+b_n\sin(nx)\right]$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{+\pi}\delta(x)\,dx=\frac{1}{\pi}\qquad a_n=\frac{1}{\pi}\int_{-\pi}^{+\pi}\delta(x)\cos(nx)\,dx=\frac{1}{\pi}\cos(nx)\Big|_{x=0}=\frac{1}{\pi}\qquad b_n=\frac{1}{\pi}\int_{-\pi}^{+\pi}\delta(x)\sin(nx)\,dx=0$$

$$f(x)=\frac{1}{2\pi}+\frac{1}{\pi}\left[\cos x+\cos(2x)+\cos(3x)+\cos(4x)\ldots\right]$$

2. Fourier Transform. $F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\delta(x-2\pi)e^{-ikx}dx=\frac{1}{\sqrt{2\pi}}e^{-ikx}\Big|_{x=2\pi}=\frac{1}{\sqrt{2\pi}}e^{-ik2\pi}$

$$F(1)=\frac{1}{\sqrt{2\pi}}e^{-i2\pi}=\frac{1}{\sqrt{2\pi}}\left[\cos(-2\pi)+i\sin(-2\pi)\right]=\frac{1}{\sqrt{2\pi}}\cos(2\pi)=\frac{1}{\sqrt{2\pi}}$$

3. Laplace Transform. $L\{t\}=\int_0^{\infty}te^{-st}dt=-\frac{d}{ds}\int_0^{\infty}e^{-st}dt=-\frac{d}{ds}\left[\frac{e^{-st}}{-s}\Bigg|_0^{\infty}\right]=-\frac{d}{ds}\left[\frac{0-1}{-s}\right]=-\frac{d}{ds}\frac{1}{s}=\frac{1}{s^2}$

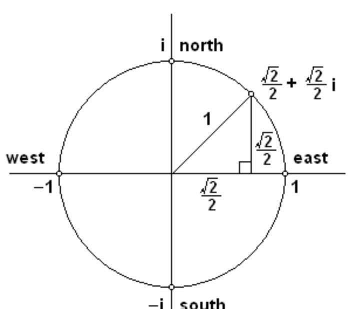
4. Laplace Transform of a Differential Equation. $\frac{dx}{dt}=at$ with $x(0)=x_0$ gives $sX(s)-x(0)=a\frac{1}{s^2}$

$X(s)=x_0\frac{1}{s}+a\frac{1}{s^3}$. Then $L\{t^n\}=\frac{n!}{s^{n+1}}$ => $L\{t^0\}=\frac{1}{s}=L\{1\}$ and $L\{t^2\}=\frac{2}{s^3}$ => $x(t)=x_0+a\frac{t^2}{2}$

5. Convolution. $\int_0^t ue^{-\lambda(t-u)}du=e^{-\lambda t}\int_0^t ue^{\lambda u}du=e^{-\lambda t}\frac{d}{d\lambda}\int_0^t e^{\lambda u}du=e^{-\lambda t}\frac{d}{d\lambda}\left[\frac{e^{\lambda u}}{\lambda}\Bigg|_0^t\right]=e^{-\lambda t}\frac{d}{d\lambda}\left[\frac{e^{\lambda t}-1}{\lambda}\right].$

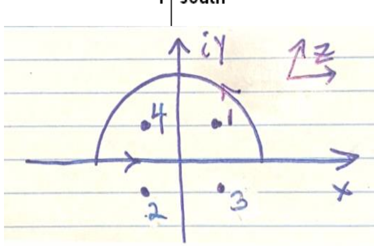
6. Cauchy-Riemann. $z^2=(x+iy)^2=x^2-y^2+2xyi=u+iv$ $\quad\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=2x\quad\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=-2y$

7. Complex Integration. $I=\int_{-\infty}^{\infty}\frac{dx}{x^4+4}\qquad I=\oint\frac{1}{z^4+4}dz\qquad I=\oint\frac{1}{(z^2+2i)(z^2-2i)}dz$



Poles: $z=\pm\sqrt{-2i}$ and $z=\pm\sqrt{2i}$. Since i is a $90°$ counterclockwise rotation in the complex plane, $\sqrt{i}$ is at $45°$ => $\sqrt{i}=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$. Since $-i$ is a $90°$ clockwise rotation, $\sqrt{-i}$ is at $-45°$ => $\sqrt{-i}=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$. Then $\pm\sqrt{2i}=\pm(1+i)$ and $\pm\sqrt{-2i}=\pm(1-i)$.

Poles: $z_1=1+i$, $z_2=-1-i$, $z_3=1-i$, $z_4=-1+i$. Only Poles 1 and 4 are inside.



$$I=\oint\frac{1}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}dz=\oint F(z)dz$$

$$F(z)=\frac{1}{(z-1-i)(z+1+i)(z-1+i)(z+1-i)}$$

$$Res(F,z_1)=\frac{1}{(1+i+1+i)(1+i-1+i)(1+i+1-i)}=\frac{1}{(2+2i)(2i)(2)}=\frac{1}{8i(1+i)}$$

$$Res(F,z_4)=\frac{1}{(-1+i-1-i)(-1+i+1+i)(-1+i-1+i)}=\frac{1}{(-2)(2i)(-2+2i)}=\frac{1}{8i(1-i)}$$

$$I=2\pi i\sum_{n=1,4}Res(F,z_n)=2\pi i\left[\frac{1}{8i(1+i)}+\frac{1}{8i(1-i)}\right]=\frac{\pi}{4}\left[\frac{1}{1+i}+\frac{1}{1-i}\right]=\frac{\pi}{4}\left[\frac{1-i+1+i}{(1+i)(1-i)}\right]=\frac{\pi}{4}\frac{2}{(1+1)}=\frac{\pi}{4}$$