Theoretical Physics | **Final Exam – Honor Code** | **May 5, 2020**
Prof. Ruiz, UNC Asheville | **Closed Everything but a Calculator**

For complete credit all steps must be shown and final answers must be in simplest forms.
For example the answer $e^{i\pi}$ can be further simplified to $e^{i\pi} = \cos\pi + i\sin\pi = -1$.

[5] Q1. Complex Numbers. Determine the cube root of i, clearly showing all steps and reasoning.

[5] Q2. Rotation. Use rotation matrices to derive the following identity, clearly showing each step.

$$\sin\frac{3\alpha}{2} = \sin\alpha\cos\frac{\alpha}{2} + \cos\alpha\sin\frac{\alpha}{2}.$$

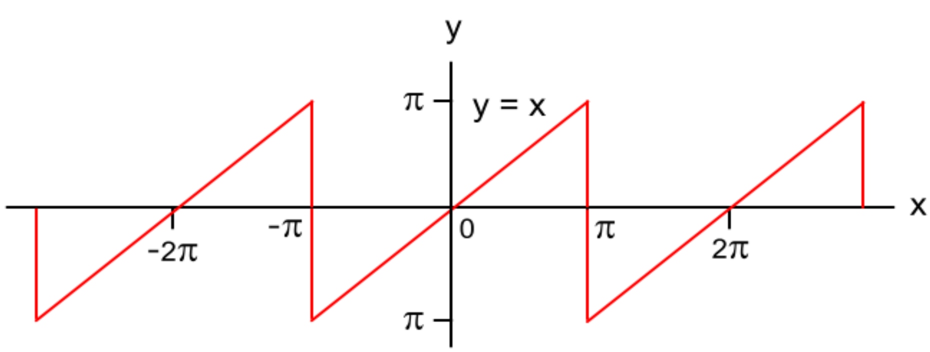
[5] Q3. Integration Derivative Trick. You know that $\int \cos(nx)dx = \frac{\sin(nx)}{n} + C_0$ since the derivative of sin(u) is cos(u). Note that C_0 means a constant. Use the derivative trick, clearly showing all steps, to arrive at

$$\int x\sin(nx)dx = \frac{\sin(nx) - nx\cos(nx)}{n^2} + C,$$ where C means a constant.

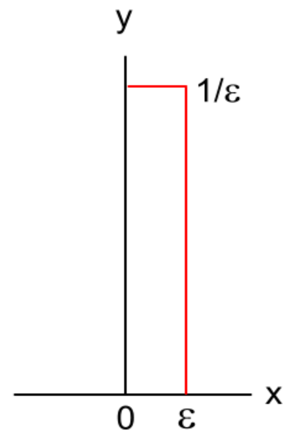
[5] Q4. Statistical Mechanics. A system consists of two energy states: $-\varepsilon$ and ε. There are 1000 particles, where each particle is in one or the other state. How many particles are expected to be in the higher-energy state if the energy $\varepsilon = kT$. You answer should be an integer. Show all steps.

[15] 4. Eigenvalues and Eigenvectors. Find the normalized eigenvectors and eigenvalues for the matrix operator $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$. A vector $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is normalized if $c_1^* c_1 + c_2^* c_2 = 1$. Show all steps.

[15] Q6. Fourier Series. Find the Fourier series for the ramp wave shown in the figure. The interval of interest is from $-\pi$ to π where $y = f(x) = x$. Note that you will need the integral derived in Q3, which means you are doing everything from scratch, like Feynman. Write out the first 5 terms of the Fourier series in simplest form. Be sure to write down the general formulas at the start, then apply them to your situation, show all steps, and write out the first 5 terms in simplest form.



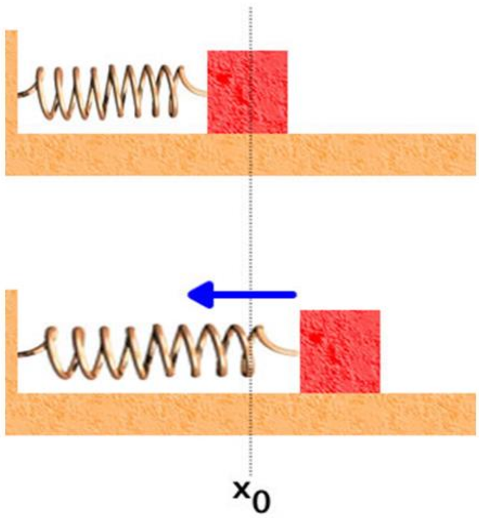
[10] Q7. Fourier Transform. Show all steps clearly for (a) and (b) below.



(a) Take the Fourier transform of the function $f(x)$, where $f(x)=\frac{1}{\varepsilon}$ in the region $0 \le x \le \varepsilon$ and $f(x)=0$ everywhere else.

(b) Show that for small ε you obtain the same result as if you had taken the Fourier transform of $\delta(x)$. You should include calculating $\Im\{\delta(x)\}$.

[15] Q8. Laplace Transform. Show all steps clearly.



Courtesy David M. Harrison

A mass attached to a spring with spring constant k is pulled to the right a distance A and let go. There is no friction. The differential equation governing the system is given by Newton's Law in conjunction with Hooke's Law.

$$m\frac{dx^2(t)}{dt^2} = -kx$$

An equivalent form is

$$\frac{dx^2(t)}{dt^2} = -\omega^2 x \text{, where } \omega = \sqrt{\frac{k}{m}}.$$

Take the Laplace transform of the differential equation and solve the algebraic equation to obtain the transform function $X(s)$. For the next step you can use the following results from the Laplace transform tables.

$$L\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2} \text{ and } L\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}.$$

From these table entries, write down the solution $x(t)$ in its simplest form.

[15] Q9. Complex Integration. Use complex integration to evaluate $\int_{-\infty}^{\infty} \frac{dx}{x^2+25}$, including a contour diagram and showing all steps clearly.

[10] Q10. Green's Function. Find the Green's function for $-\frac{d^2x}{dt^2} - \frac{dx}{dt} + 6x = f(t)$, showing all steps clearly and including a contour diagram.

Theoretical Physics/Ruiz **Final Exam - Solutions** **May 5, 2020**

[5] 1. Complex Numbers. $\sqrt[3]{i} = (e^{i\frac{\pi}{2}})^{\frac{1}{3}} = e^{i\frac{\pi}{6}} = \cos 30° + i \sin 30° = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

[5] 2. Rotation Matrix. $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\frac{\alpha}{2} & \sin\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{bmatrix} = \begin{bmatrix} \cos\frac{3\alpha}{2} & \sin\frac{3\alpha}{2} \\ -\sin\frac{3\alpha}{2} & \cos\frac{3\alpha}{2} \end{bmatrix}$

$$\sin\frac{3\alpha}{2} = \cos\alpha \sin\frac{\alpha}{2} + \sin\alpha\cos\frac{\alpha}{2} = \sin\alpha\cos\frac{\alpha}{2} + \cos\alpha\sin\frac{\alpha}{2}.$$

[5] Q3. Derivative Trick. $\int \cos(nx)dx = \frac{\sin(nx)}{n} + C_0$, where C_0 means a constant.

$$\int x\sin(nx)dx = -\frac{d}{dn}\left[\int \cos(nx)dx\right] + C = -\frac{d}{dn}\left[\frac{\sin(nx)}{n} + C_0\right] + C$$

$$\int x\sin(nx)dx = -\frac{n\cos(nx)}{n} + \frac{\sin(nx)}{n^2} + C = \frac{\sin(nx) - nx\cos(nx)}{n^2} + C$$

[5] Q4. Statistical Mechanics. A system of N = 1000 particles in either state $-\varepsilon$ or ε.

$n_\varepsilon = \frac{Ne^{-\frac{\varepsilon}{kT}}}{Z}$ with $Z = e^{+\frac{\varepsilon}{kT}} + e^{-\frac{\varepsilon}{kT}}$. Then $n_{\varepsilon=kT} = \frac{1000 \cdot e^{-1}}{e + e^{-1}} = 119$.

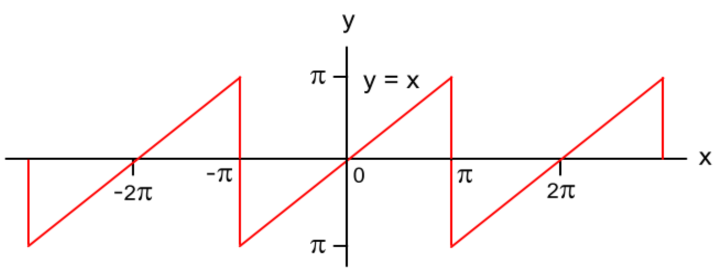
[15] 4. Eigenvalues and Eigenvectors. $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $\det\begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix} = 0$, $\lambda^2 + 3\lambda + 2 = 0$

$(\lambda+1)(\lambda+2) = 0$ gives eigenvalues -1 and -2.

$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ gives $c_2 = -c_1$, normalized eigenvector $u = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -2\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ gives $c_2 = -2c_1$, leading to $v = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

[15] Q6. Fourier Series. Odd function, so we need $b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx$

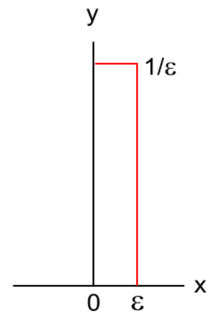


$$b_n = \frac{2}{\pi}\left[\frac{\sin(nx) - nx\cos(nx)}{n^2}\right]_0^{\pi}$$

$$b_n = \frac{2}{\pi}\left[\frac{-x\cos(nx)}{n}\right]_0^{\pi} = -\frac{2}{\pi}\left[\frac{\pi}{n}\cos(n\pi) - 0\right]$$

$b_n = -\frac{2}{n}(-1)^n$ $\quad f(x) = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin(nx)$ $\quad f(x) = 2\left[\sin(nx) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \frac{\sin(4x)}{4} + \frac{\sin(5x)}{5}...\right]$

[10] Q7. Fourier Transform.



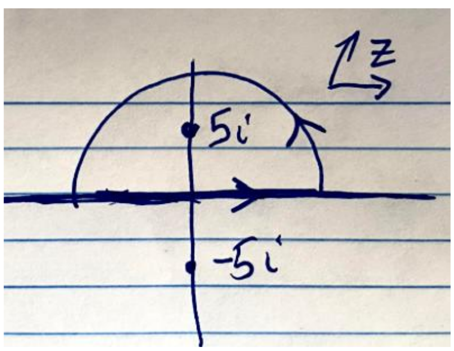
$$F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx=\frac{1}{\sqrt{2\pi}}\int_{0}^{\varepsilon}\frac{1}{\varepsilon}e^{-ikx}dx$$

$$F(k)=\frac{1}{\sqrt{2\pi}}\frac{1}{\varepsilon}\left[\frac{e^{-ikx}}{-ik}\right]_0^{\varepsilon}=\frac{1}{\sqrt{2\pi}}\frac{1}{\varepsilon}\frac{e^{-ik\varepsilon}-1}{-ik}=\frac{1}{\sqrt{2\pi}}\frac{1-e^{-ik\varepsilon}}{ik\varepsilon}$$

For small ε ,

$$F(k)\approx\frac{1}{\sqrt{2\pi}}\frac{[1-(1-ik\varepsilon)]}{ik\varepsilon}=\frac{1}{\sqrt{2\pi}}\frac{ik\varepsilon}{ik\varepsilon}=\frac{1}{\sqrt{2\pi}}=\Im\{\delta(x)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x)e^{-ikx}dx\,.$$

[15] Q8. Laplace Transform. $L\{\frac{dx^2(t)}{dt^2}\}=L\{-\omega^2 x\}$ gives $s^2X(s)-sx(0)-x'(0)=-\omega^2X(s)$, where for our problem $x(0)=A$ and $x'(0)=v(0)=0$. Therefore, $s^2X(s)-sA=-\omega^2X(s)$, $(s^2+\omega^2)X(s)=sA$, and $X(s)=A\frac{s}{s^2+\omega^2}$. From the table given: $x(t)=A\cos(\omega t)$.



[15] Q9. Complex Integration.

$$\int_{-\infty}^{\infty}\frac{dx}{x^2+25}=\oint\frac{dz}{z^2+25}=\oint\frac{dz}{(z+5i)(z-5i)}=\oint F(z)dz$$

$$\int_{-\infty}^{\infty}\frac{dx}{x^2+25}=2\pi i\mathrm{Res}\left[F(z),5i\right]$$

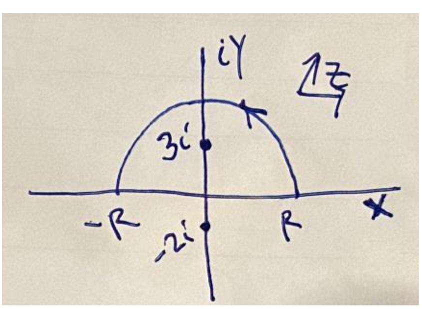
$$\int_{-\infty}^{\infty}\frac{dx}{x^2+25}=2\pi i\left.\frac{1}{z+5i}\right|_{z=5i}=2\pi i\frac{1}{5i+5i}=2\pi i\frac{1}{10i}=\frac{\pi}{5}$$

[10] Q10. Green's Function. Find the Green's function for $-\frac{d^2x}{dt^2}-\frac{dx}{dt}+6x=f(t)$

Delta: $-\frac{d^2x}{dt^2}-\frac{dx}{dt}+6x=\delta(t)$. Fourier: $-(i\omega)^2X(\omega)-i\omega X(\omega)+6X(\omega)=\frac{1}{\sqrt{2\pi}}$

$$X(\omega)=\frac{1}{\sqrt{2\pi}}\frac{1}{\omega^2-i\omega+6}=\frac{1}{\sqrt{2\pi}}\frac{1}{(\omega+2i)(\omega-3i)}$$

Inverse Transform: $x(t)=\Im^{-1}\{X(\omega)\}\equiv G(t)$



$$G(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\frac{e^{i\omega t}}{(\omega+2i)(\omega-3i)}d\omega$$

$$G(t)=\frac{1}{2\pi}\oint\frac{e^{izt}}{(z+2i)(z-3i)}dz$$

$$G(t)=\frac{1}{2\pi}2\pi iRes\left[\frac{e^{izt}}{(z+2i)(z-3i)},3i\right]=\frac{1}{2\pi}2\pi i\left.\frac{e^{izt}}{(z+2i)}\right|_{z=3i}=\frac{e^{-3t}}{5}=G(t,0)$$