Theoretical Physics (Prof. Ruiz/UNCA) Exam 1 February 22, 2021

Open Book/Notes/Internet (90 minutes including scan and email return).
All work and related steps must be explicitly shown for full credit.

[10] 1. Groups. A group $G = \{a, b, c, d, e\}$ with identity e. For any $a \in G$ and $b \in G$, the following is true: $aba^{-1}b^{-1} = e$. Prove that the group is abelian.

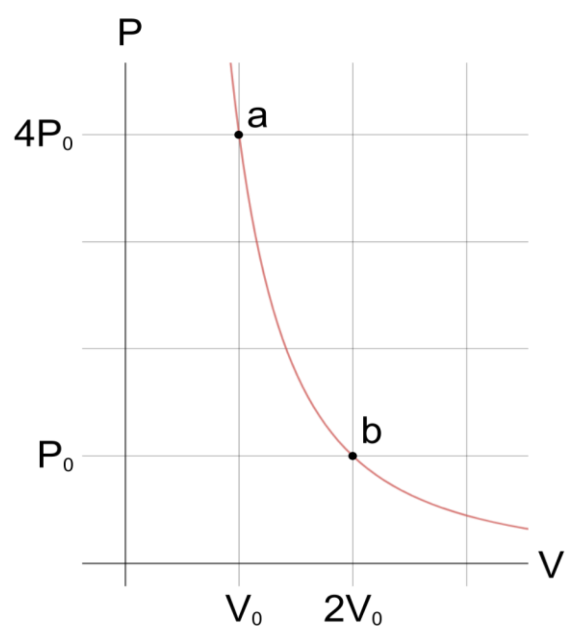
[25] 2. Integral. Use a derivative trick to evaluate $\int_0^\infty xe^{-ax}\sin(kx)\,dx$, where $a > 0$, starting from the integral result $\int_0^\infty e^{-ax}\cos(kx)\,dx = \frac{a}{a^2 + k^2}$. For 20 points max, you may instead evaluate $\int_0^\infty xe^{-ax}\cos(kx)\,dx$. Commit to one for official credit.

[25] 3. Waves. Show that $\psi(x,t) = Ae^{-i(ax+bt)}$ satisfies the wave equation with an associated auxiliary equation that relates a, b, and the velocity v. What is this equation, in simplest form, that relates a, b, and v?

[10] 4. E&M. If the second Maxwell equation was modified to be $\nabla \cdot \vec{B} = f(x, y, z) \neq 0$, explain in a sentence or two the new physics of this novel situation.

[30] 5. Gas and Work. A gas expands from point a to point b (see figure), where the pressure varies as $P = \frac{k}{V^2}$ during the expansion (k is a constant). What is the work done by the gas if the gas expands from $V_1 = V_0$ to $V_2 = 2V_0$ as shown in the figure, i.e., from point a to point b?

Give your answer in terms of P_0 and V_0, where k does not appear in your answer.



1. Groups. Given: $aba^{-1}b^{-1} = e$. Multiply both sides by b on the right. Then $aba^{-1}b^{-1}b = eb$ => $aba^{-1}e = b$ => $aba^{-1} = b$. Multiply both sides by a on the right to get $aba^{-1}a = ba$, which leads to $abe = ba$ and finally $ab = ba$ (abelian).

2. Integral. $\int_0^\infty xe^{-ax}\sin(kx)\,dx$, where $a > 0$ using a derivative.

$$\int_0^\infty xe^{-ax}\sin(kx)\,dx = -\frac{d}{dk}\int_0^\infty e^{-ax}\cos(kx)\,dx = -\frac{d}{dk}\left(\frac{a}{a^2+k^2}\right)$$

$$\int_0^\infty xe^{-ax}\sin(kx)\,dx = -\left[-\frac{a}{(a^2+k^2)^2}\right]2k = \frac{2ak}{(a^2+k^2)^2}$$

3. Waves. The function $\psi(x,t) = Ae^{-i(ax+bt)}$ => $\dfrac{\partial\psi}{\partial x} = -iaAe^{-i(ax+bt)} = -ia\psi$

$$\frac{\partial^2\psi}{\partial x^2} = (-ia)^2\psi = -a^2\psi \qquad \frac{\partial^2\psi}{\partial t^2} = (-ib)^2\psi = -b^2\psi$$

Wave equation $\dfrac{\partial^2\psi}{\partial x^2} = \dfrac{1}{v^2}\dfrac{\partial^2\psi}{\partial t^2}$ is satisfied if $a^2 = \dfrac{b^2}{v^2}$ => $v = \pm\dfrac{b}{a}$.

4. E&M. Given $\nabla\cdot\vec{B} = f(x,y,z) \neq 0$ would imply that magnetic charge exists, i.e., magnetic monopoles. Note that $\nabla\cdot\vec{E} = \dfrac{\rho}{\varepsilon_0} \neq 0$ implies the existence of electric charge.

5. Gas and Work. Work $W = \displaystyle\int_{V_1}^{V_2} PdV = \int_{V_1}^{V_2}\frac{k}{V^2}dV = -k\,\frac{1}{V}\bigg|_{V1}^{V_2} = -k\,\frac{1}{V}\bigg|_{V_0}^{2V_0}$

$$W = -k\left(\frac{1}{2V_0} - \frac{1}{V_0}\right) = \frac{k}{V_0}\left(1 - \frac{1}{2}\right) = \frac{k}{2V_0}$$

Note that $4P_0 = \dfrac{k}{V_0^2}$.

Use $k = 4P_0V_0^2$ to obtain $W = \dfrac{k}{2V_0} = \dfrac{4P_0V_0^2}{2V_0} = 2P_0V_0$.

APPENDIX (Alternative Choice for Problem 2).

$$\int_0^\infty xe^{-ax}\cos(kx)\,dx = -\frac{d}{da}\int_0^\infty e^{-ax}\cos(kx)\,dx$$

$$\int_0^\infty xe^{-ax}\cos(kx)\,dx = -\frac{d}{da}\left(\frac{a}{a^2+k^2}\right)$$. Use the product rule.

$$\int_0^\infty xe^{-ax}\cos(kx)\,dx = -\frac{1}{a^2+k^2} - (-1)\frac{a}{(a^2+k^2)^2}\,2a$$

$$\int_0^\infty xe^{-ax}\cos(kx)\,dx = -\frac{1}{a^2+k^2} + \frac{2a^2}{(a^2+k^2)^2}$$

$$\int_0^\infty xe^{-ax}\cos(kx)\,dx = \frac{-(a^2+k^2)+2a^2}{(a^2+k^2)^2} = \frac{a^2-k^2}{(a^2+k^2)^2}$$