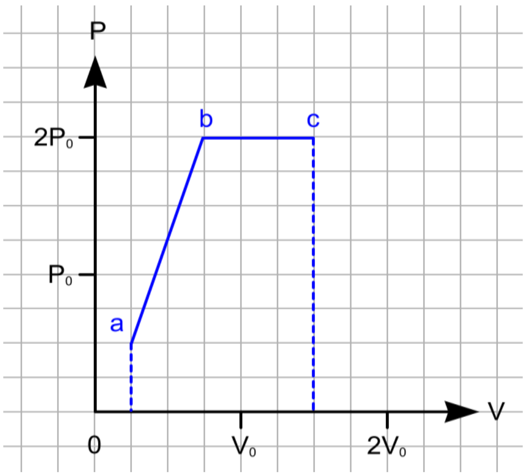
Theoretical – Ruiz/UNCA Final Exam (6:15 pm - 9:30 pm) – Open Everything May 3, 2021

[15] Q1. Taylor Series. Show that $f''(x) = -\dfrac{1}{4f^3}$ and $f'''(x) = \dfrac{3}{8f^5}$, given $f'(x) = \dfrac{1}{2f}$.

Then, given that $f(4) = 2$, find the first four terms of the Taylor series expansion for $f(x)$ about the point $x = 4$. Evaluate $f(4.1)$ from the first two terms of your series to 5 significant figures.

[10] Q2. Integration with the Derivative Trick. The integral $\int_{-\infty}^{\infty} f(x) e^{-ax^2} dx = g(x,a)$. How would you set things up to evaluate $\int_{-\infty}^{\infty} x^6 f(x) e^{-ax^2} dx$ using the derivative trick.



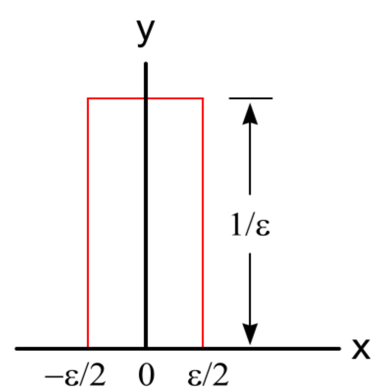
[10] Q3. Work. Calculate the work exactly in terms of integers, P_0, and V_0 for a system that undergoes an ideal path from point a to b and then to c in the PV diagram at the left.

[15] Q4. Eigenvectors and Eigenvalues. Find the normalized eigenvectors and eigenvalues for the matrix operator

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}.$$

For all Questions on this Final, show all details for full credit.

[15] Q5. Fourier Transform. (a) Take the Fourier transform of the function $f(x)$, where $f(x) = \dfrac{1}{\varepsilon}$ for $-\varepsilon/2 \le x \le \varepsilon/2$ and $f(x) = 0$ elsewhere. (b) Show that for small ε you obtain the same result as the Fourier transform of $\delta(x)$. You should include calculating $\Im\{\delta(x)\}$.



[15] 6. Convolution. Sketch $f(t) = 1$ for $0 \le t \le 1$ and zero elsewhere. Then sketch $g(t) = t$ for $0 \le t \le 1$ and zero elsewhere. Find the convolution $h(t) = f * g$ inside the nonzero region $0 \le t \le 1$. To gain insight into the convolution, imagine the square pulse moving in from the far left and overlapping your ramp as the square pulse travels to the right. Sketch an overlap where the right edge of the square pulse is at some arbitrary t in the region $0 \le t \le 1$. Your left rectangular edge will be at $t - 1$. What is the area of the common overlapped region?

[5] Q7. Complex Integration. Use complex integration to evaluate $\int_{-\infty}^{\infty} \dfrac{dx}{x^2 + 49}$, including a contour diagram and showing all steps clearly.

[15] Q8. Green's Function. Find the Green's function for $-\dfrac{d^2x}{dt^2} - \dfrac{dx}{dt} + 2x = f(t)$.

Q1. Taylor Series. Taylor series about the point $x = 4$.

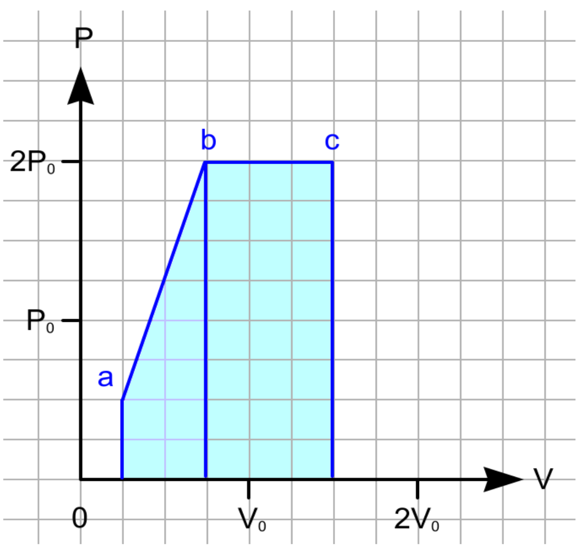
$f' = \frac{f^{-1}}{2}$	$f'' = \frac{(-f^{-2})}{2} f' = -\frac{f^{-2}}{2}\frac{f^{-1}}{2} = -\frac{f^{-3}}{4} = -\frac{1}{4f^3}$	$f''' = \frac{3f^{-4}}{4} f' = \frac{3f^{-5}}{8} = \frac{3}{8f^5}$

$f(4) = 2$	$f'(4) = \frac{1}{2f(4)} = \frac{1}{2\cdot 2} = \frac{1}{4}$	$f''(4) = -\frac{1}{4f^3(4)} = -\frac{1}{32}$	$f'''(4) = \frac{3}{8f^5(4)} = \frac{3}{8\cdot 32} = \frac{3}{256}$

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(4)\frac{(x-4)^n}{n!} = 2\frac{(x-4)^0}{0!} + \frac{1}{4}\frac{(x-4)^1}{1!} - \frac{1}{32}\frac{(x-4)^2}{2!} + \frac{3}{256}\frac{(x-4)^3}{3!} + \ldots$$

$$f(x) = 2 + \frac{x-4}{4} - \frac{(x-4)^2}{64} + \frac{(x-4)^3}{512} + \ldots \qquad f(4.1) = 2 + \frac{4.1-4}{4} = 2 + \frac{0.1}{4} = 2.0250$$

Q2. $$\int_{-\infty}^{\infty} x^6 f(x) e^{-ax^2} dx = \left[-\frac{d}{da}\right]^3 \int_{-\infty}^{\infty} f(x) e^{-ax^2} dx = \left[-\frac{d}{da}\right]^3 g(x,a)$$



Q3.

$$W_{ab} = \frac{1}{2}\left(\frac{P_0}{2} + 2P_0\right)\left(\frac{3V_0}{4} - \frac{V_0}{4}\right) = \frac{1}{2}\frac{5}{2}P_0\frac{2}{4}V_0 = \frac{5}{8}P_0V_0$$

$$W_{bc} = (2P_0)\left(\frac{3}{2}V_0 - \frac{3}{4}V_0\right) = 2P_0\frac{3}{4}V_0 = \frac{3}{2}P_0V_0$$

$$W_{ac} = \left(\frac{5}{8} + \frac{3}{2}\right)P_0V_0 = \left(\frac{5+12}{8}\right)P_0V_0 = \frac{17}{8}P_0V_0$$

Q4. $$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \Rightarrow \begin{vmatrix} 1-\lambda & 0 \\ -1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda) = 0 \Rightarrow \lambda = 1 \;\&\; \lambda = 2$$

For $\lambda = 1$, $$\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1\cdot c_1 + 0\cdot c_2 \\ -1\cdot c_1 + 2\cdot c_2 \end{bmatrix} = 1\cdot\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Rightarrow c_1 = c_2 \Rightarrow u = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

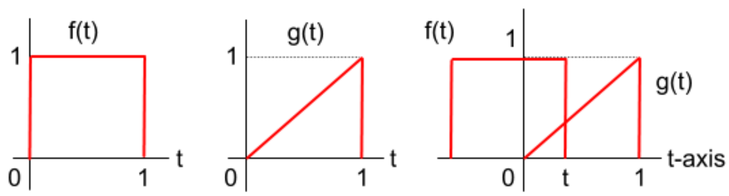
For $\lambda = 2$, $$\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1\cdot c_1 + 0\cdot c_2 \\ -1\cdot c_1 + 2\cdot c_2 \end{bmatrix} = 2\cdot\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Rightarrow c_1 = 2c_1 \Rightarrow \Rightarrow c_1 = 0 \Rightarrow v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Q5. $$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}}\int_{-\varepsilon/2}^{\varepsilon/2} \frac{1}{\varepsilon} e^{-ikx} dx = \frac{1}{\sqrt{2\pi}}\frac{1}{\varepsilon}\left[\frac{e^{-ikx}}{-ik}\right]_{-\varepsilon/2}^{\varepsilon/2}$$

$$F(k)=\frac{1}{\sqrt{2\pi}}\frac{1}{\varepsilon}\frac{e^{-ik\varepsilon/2}-e^{ik\varepsilon/2}}{(-ik)}=\frac{1}{\sqrt{2\pi}}\frac{e^{ik\varepsilon/2}-e^{-ik\varepsilon/2}}{ik\varepsilon}=\frac{2\sin(k\varepsilon/2)}{\sqrt{2\pi}k\varepsilon}=\frac{\operatorname{sinc}(k\varepsilon/2)}{\sqrt{2\pi}}$$

$$F(k)=\frac{\operatorname{sinc}(k\varepsilon/2)}{\sqrt{2\pi}}\underset{\varepsilon\to 0}{\to}\frac{1}{\sqrt{2\pi}}\quad \Im\{\delta(x)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x)e^{-ikx}dx=\frac{1}{\sqrt{2\pi}}e^{-ikx}\Big|_{x=0}=\frac{1}{\sqrt{2\pi}}$$

Q6.

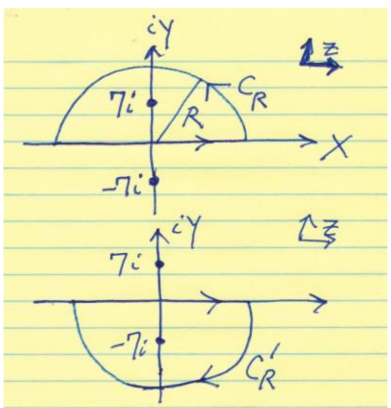


$$f(t)*g(t)=\int_0^t f(u)g(t-u)du$$

Overlap area $\frac{1}{2}(t)(t)=\frac{t^2}{2}$.

$$h(t)=f(t)*g(t)=\int_0^t 1\cdot(t-u)du=(tu-\frac{u^2}{2})\Big|_0^t=t^2-\frac{t^2}{2}=\frac{t^2}{2}$$. It is the overlap area.

Q7. For this contour, you have a choice: upper or lower, but NOT both. Why must you decide on one?



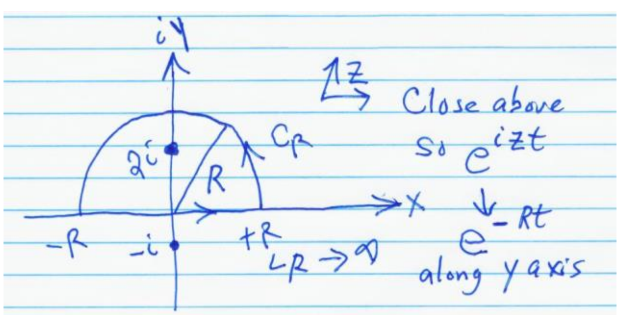
$$\int_{-\infty}^{\infty}\frac{dx}{x^2+49}=\oint\frac{1}{(z+7i)(z-7i)}dz=2\pi i Res\left[\frac{1}{(z+7i)(z-7i)},7i\right]$$

$$\int_{-\infty}^{\infty}\frac{dx}{x^2+49}=2\pi i\frac{1}{(z+7i)}\Big|_{z=7i}=\frac{2\pi i}{14i}=\frac{\pi}{7}$$ Why does upper work?

$$\int_{-\infty}^{\infty}\frac{dx}{x^2+49}=\oint\frac{1}{(z+7i)(z-7i)}dz=-2\pi i Res\left[\frac{1}{(z+7i)(z-7i)},-7i\right]$$

$$\int_{-\infty}^{\infty}\frac{dx}{x^2+49}=-2\pi i\frac{1}{(z-7i)}\Big|_{z=-7i}=\frac{-2\pi i}{-14i}=\frac{\pi}{7}$$ Why does lower work?

Q8. Given $-\frac{d^2x}{dt^2}-\frac{dx}{dt}+2x=\delta(t)$. Take the Fourier Transform of both sides.



$$-(i\omega)^2X(\omega)-i\omega X(\omega)+2X(\omega)=\frac{1}{\sqrt{2\pi}} \Rightarrow$$

$$X(\omega)=\frac{1}{\sqrt{2\pi}}\frac{1}{\omega^2-i\omega+2}=\frac{1}{\sqrt{2\pi}}\frac{1}{(\omega+i)(\omega-2i)}$$

Inverse Fourier Transform:

$$x(t)=\Im^{-1}\{X(\omega)\}\equiv G(t)$$

$$G(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\frac{e^{i\omega t}}{(\omega+i)(\omega-2i)}d\omega \qquad G(t)=\frac{1}{2\pi}\oint\frac{e^{izt}}{(z+i)(z-2i)}dz$$

$$G(t)=\frac{1}{2\pi}2\pi i Res\left[\frac{e^{izt}}{(z+i)(z-2i)},2i\right]=\frac{1}{2\pi}2\pi i\frac{e^{izt}}{(z+i)}\Big|_{z=2i}=\frac{e^{-2t}}{3}=G(t,0)$$